Chapter 7:

Fuzzy Systems

Fuzzy Set Theory

- The classical set theory is based on the term "*is element of*" ($\in$).
  Alternatively one can describe the membership
  to a set with an *indicator function*:
  Let X be a set. Then

  $$I_M : X \to \{0,1\}, \qquad I_M(x) = \begin{cases} 1, & \text{if } x \in X, \\ 0, & \text{otherwise,} \end{cases}$$

  is called **indicator function** of the set M w.r.t. the basic set X.

- In fuzzy set theory the indicator function is replaced
  by a *membership function*:
  Let X be a (classical/crisp) set. Then

  $$\mu_M : X \to [0,1], \qquad \mu_M(x) \mathrel{\widehat{=}} \text{membership degree of } x \text{ to } M,$$

  membership function of the **fuzzy set** M
  w.r.t. the *basic set* X.

  Most of the time the fuzzy set is identified by its membership function.

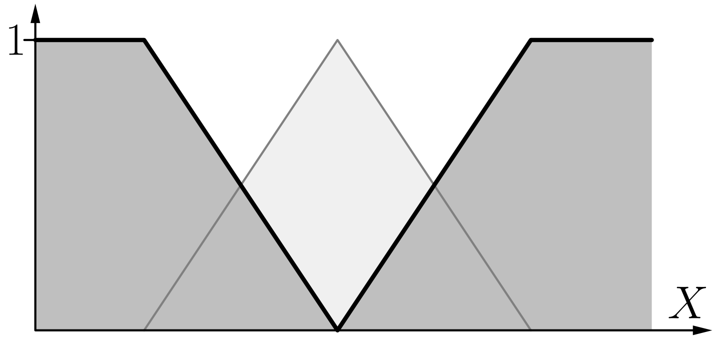
- As with the transition from the classical logic to fuzzy logic an extension of the operators is necessary for the transition from classical set theory to fuzzy set theory.

- **Basic principles of this extension**:
  Refer to the logical definitions of the operators.
  $\Rightarrow$ elementwise application of the logical operators

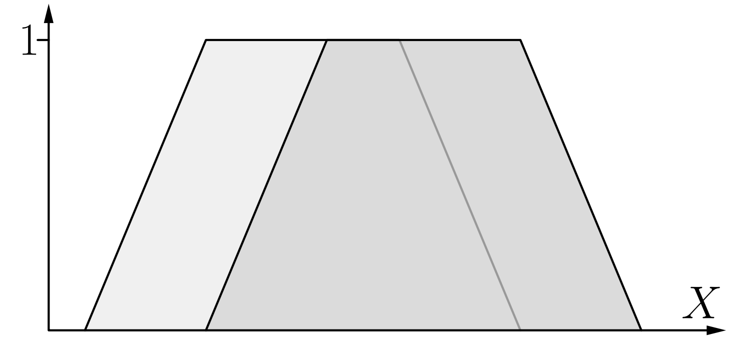
- Let A and B be (fuzzy) set over the basic set X.

Complement	classical	$\overline{A} = \{x \in X \mid x \notin A\}$
	fuzzy	$\forall x \in X: \quad \mu_{\overline{A}}(x) = \sim\mu_A(x)$
Intersection	classical	$A \cap B = \{x \in X \mid x \in A \wedge x \in B\}$
	fuzzy	$\forall x \in X: \quad \mu_{A \cap B}(x) = \top(\mu_A(x), \mu_B(x))$
Union	classical	$A \cup B = \{x \in X \mid x \in A \vee x \in B\}$
	fuzzy	$\forall x \in X: \quad \mu_{A \cup B}(x) = \bot(\mu_A(x), \mu_B(x))$

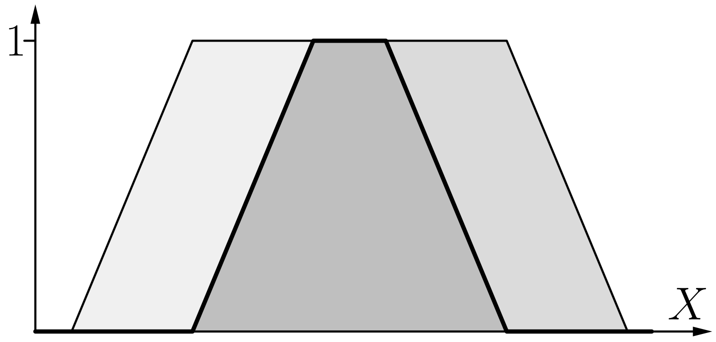
Fuzzy Set Operators: Examples



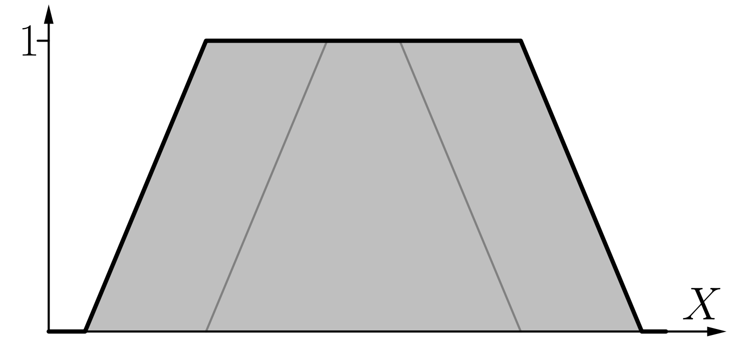
fuzzy complement



two fuzzy sets



fuzzy intersection



fuzzy union

- The fuzzy intersection shown on the left and the fuzzy union on the right are independent of the underlying t-norm and t-conorm, respectively.

Fuzzy Partitions and Linguistic Variables

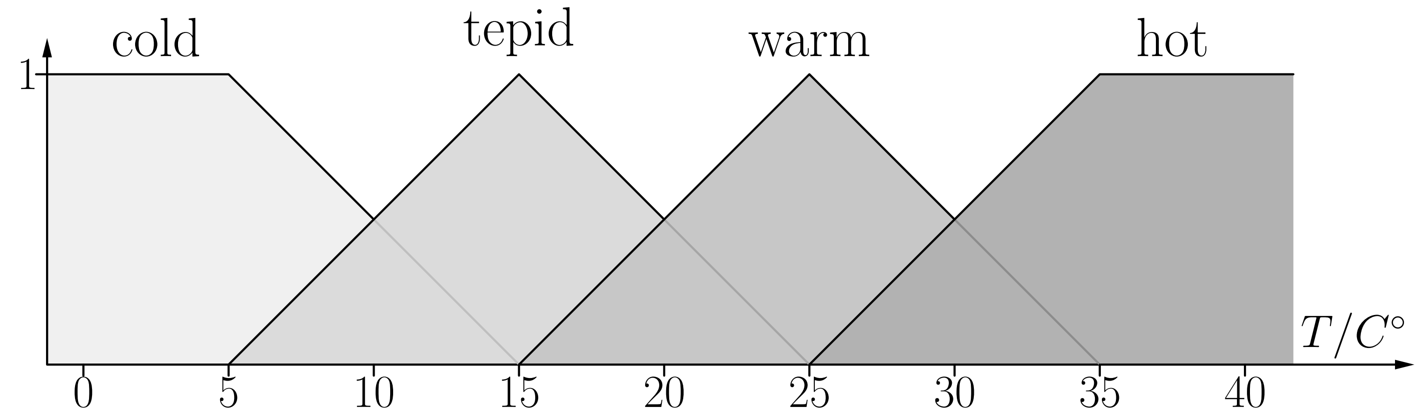
- To describe a domain with linguistic terms it is fuzzy-partitioned with a collection of fuzzy sets.

  Every fuzzy set of the partition gets assigned a linguistic term.

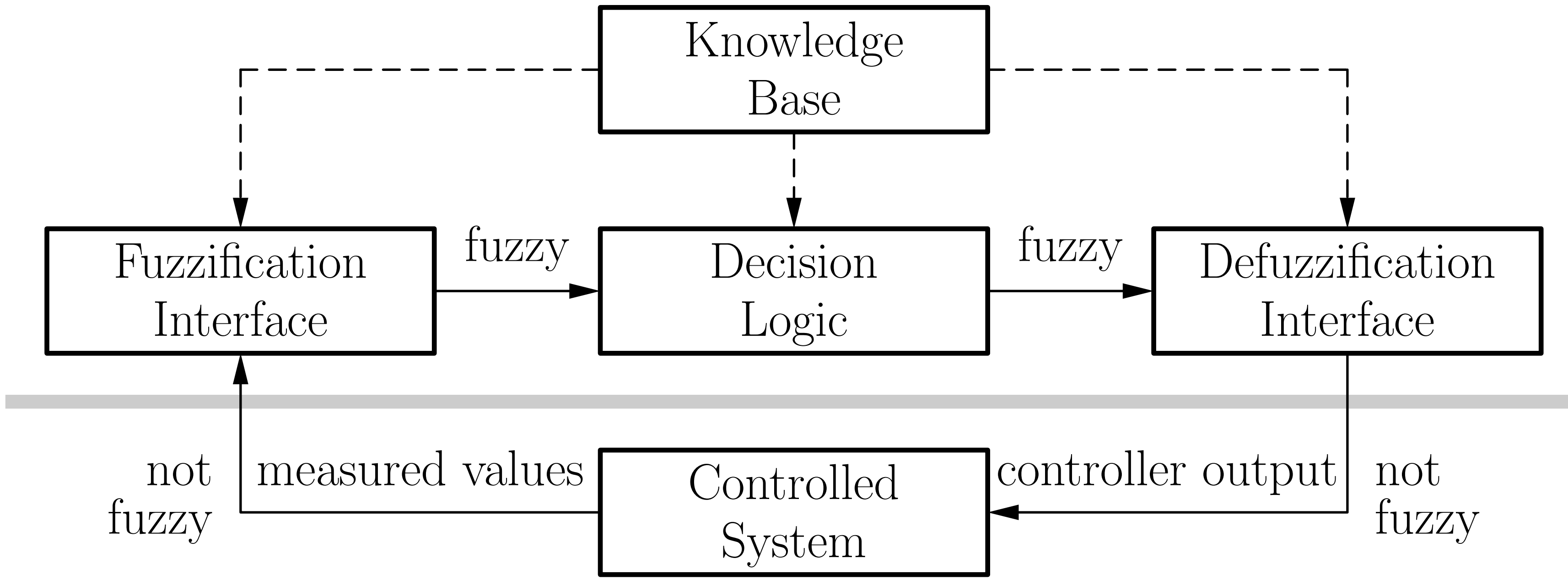
- Usual condition: at every point of the domain the membership degrees must sum up to 1 *(partition of unity)*.

Example: Fuzzy partition for temperatures

We define a linguistic variable with values *cold*, *tepid*, *warm* und *hot*.



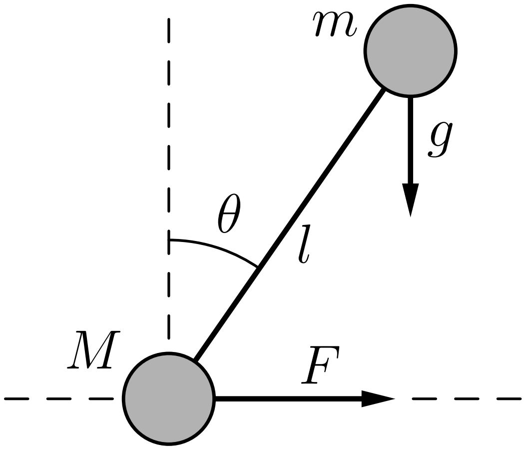
Architecture of a Fuzzy Controller

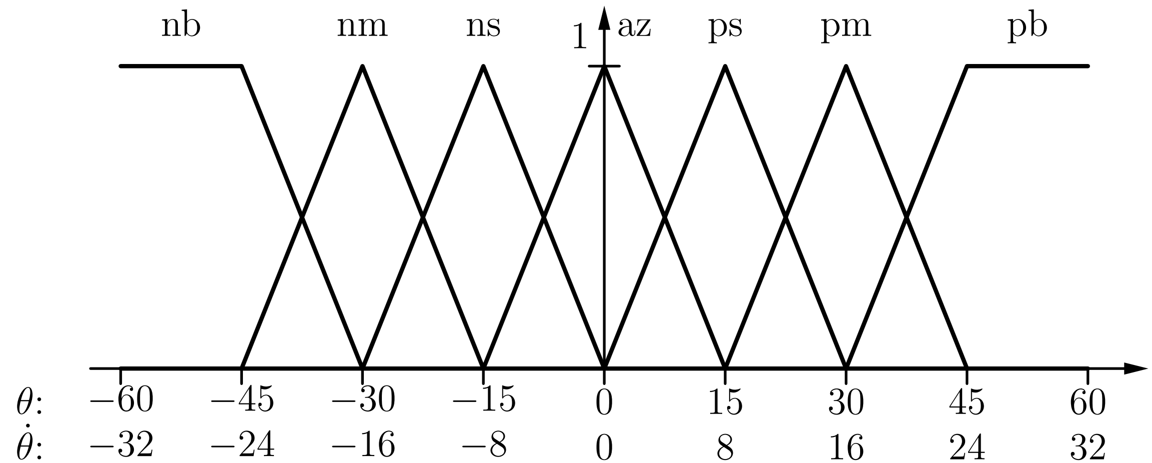


- The knowledge base contains the fuzzy rules for the controller and the fuzzy partitions of the variables' domains.
- A fuzzy rule reads: **if** X_1 **is** $A^{(1)}_{i_1}$ **and** ... **and** X_n **is** $A^{(n)}_{i_n}$ **then** Y **is** B.

  $X_1, \ldots, X_n$ are measured values and Y is the control variable.

  $A^{(k)}_{i_k}$ and B are linguistic terms with assigned fuzzy sets.



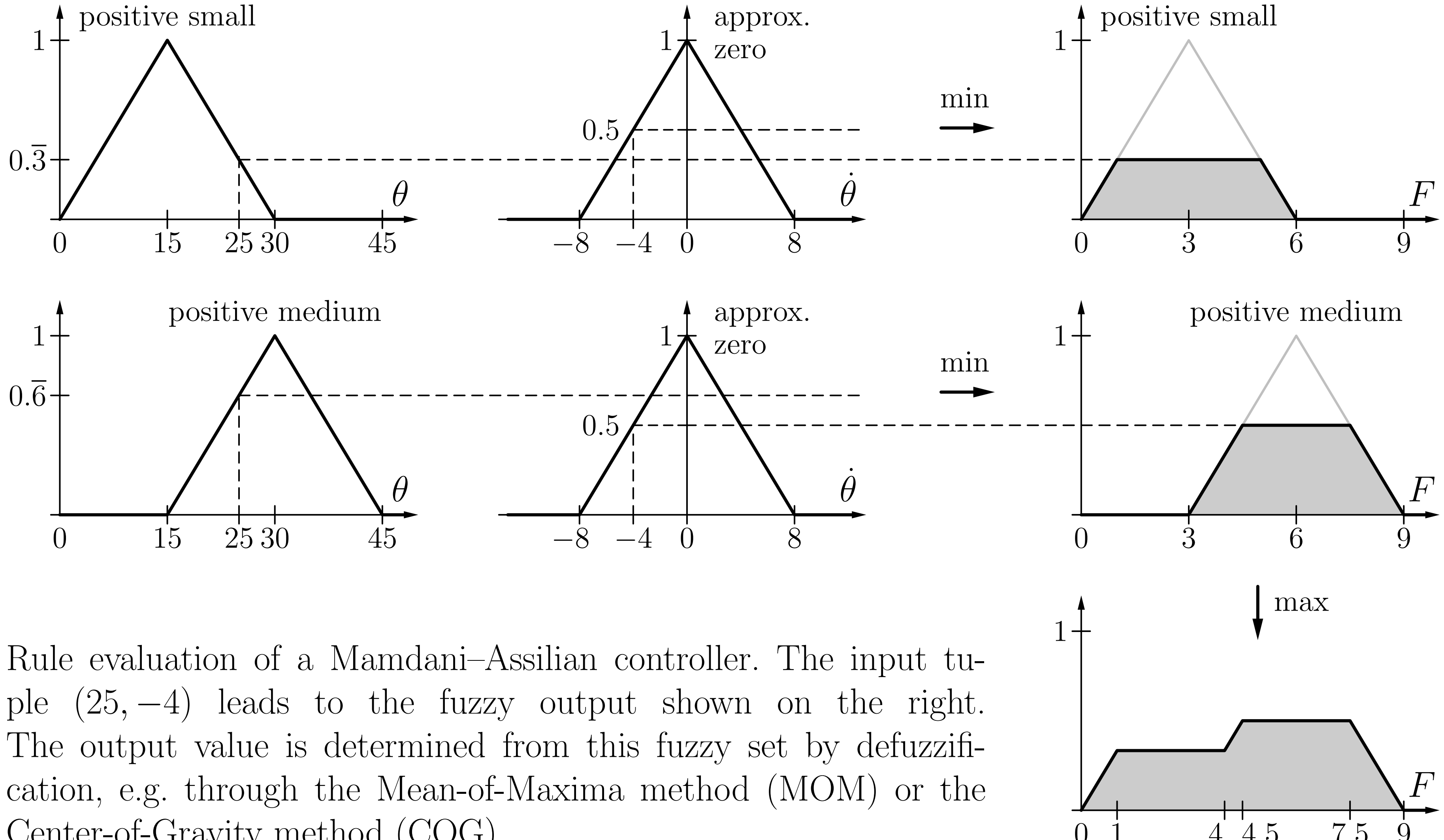


Abbreviations

pb – positive big
pm – positive medium
ps – positive small
az – approximately zero
ns – negative small
nm – negative medium
nb – negative big

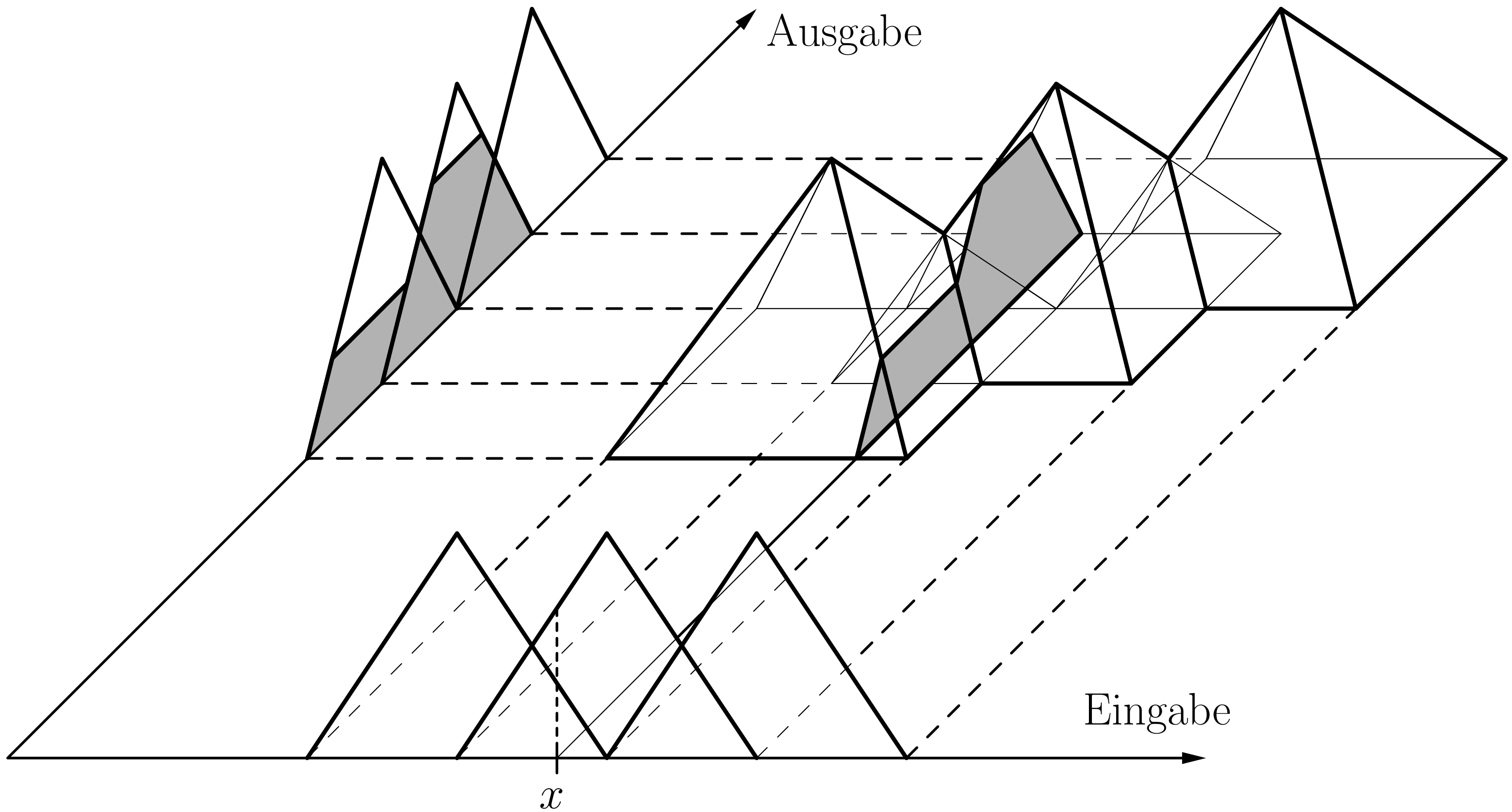
$\dot{\theta}\backslash\theta$	nb	nm	ns	az	ps	pm	pb
pb			**ps**	**pb**			
pm				**pm**			
ps	**nm**		**az**	**ps**			
az	**nb**	**nm**	**ns**	**az**	**ps**	**pm**	**pb**
ns				**ns**	**az**		**pm**
nm				**nm**			
nb				**nb**	**ns**		

Fuzzy Controller by Mamdani–Assilian



Rule evaluation of a Mamdani–Assilian controller. The input tuple $(25, -4)$ leads to the fuzzy output shown on the right. The output value is determined from this fuzzy set by defuzzification, e.g. through the Mean-of-Maxima method (MOM) or the Center-of-Gravity method (COG).

Fuzzy Controller by Mamdani–Assilian



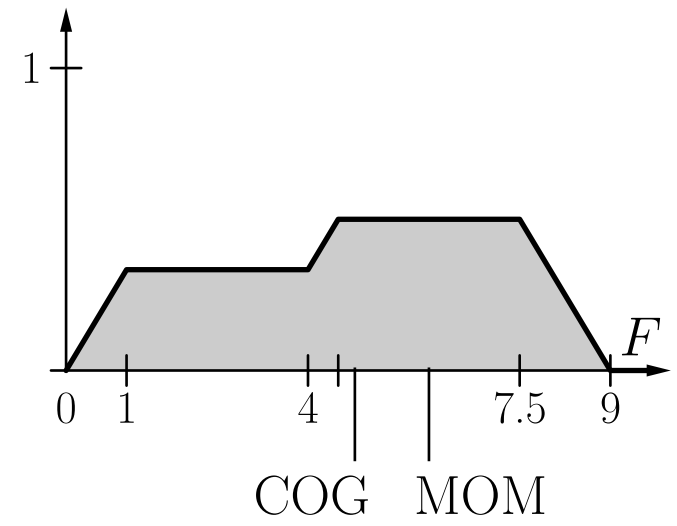
A fuzzy rule system with one input and one output variable and three fuzzy rules.
Every pyramid is defined by a fuzzy rule.
The input value x leads to the fuzzy output shaded in gray.

Defuzzification

The evaluation of the fuzzy rules results in an **output fuzzy set**.

This output fuzzy set has to be transformed into a **crisp control value**.

This task is called **defuzzification**.



The most important defuzzification methods are:

- **Center of Gravity** (COG)
  The center of gravity of the area under the output fuzzy set.
- **Center of Area** (COA)
  The point that divides the area under the output fuzzy set into equally sized parts.
- **Mean of Maxima** (MOM)
  The arithmetic mean of the locations with maximal membership degree.