Chapter 7:
Fuzzy Systems
The classical set theory is based on the term “is element of” (\(\in\)). Alternatively one can describe the membership to a set with an *indicator function*:

Let \(X\) be a set. Then

\[
I_M : X \rightarrow \{0, 1\}, \quad I_M(x) = \begin{cases} 
1, & \text{if } x \in X, \\
0, & \text{otherwise},
\end{cases}
\]

is called **indicator function** of the set \(M\) w.r.t. the basic set \(X\).

In fuzzy set theory the indicator function is replaced by a *membership function*:

Let \(X\) be a (classical/crisp) set. Then

\[
\mu_M : X \rightarrow [0, 1], \quad \mu_M(x) \equiv \text{membership degree of } x \text{ to } M,
\]

**membership function** of the **fuzzy set** \(M\) w.r.t. the **basic set** \(X\).

Most of the time the fuzzy set is identified by its membership function.
As with the transition from the classical logic to fuzzy logic, an extension of the operators is necessary for the transition from classical set theory to fuzzy set theory.

**Basic principles of this extension:**
Refer to the logical definitions of the operators.
⇒ elementwise application of the logical operators

Let $A$ and $B$ be (fuzzy) set over the basic set $X$.

<table>
<thead>
<tr>
<th>Complement</th>
<th>classical</th>
<th>$\overline{A} = {x \in X \mid x \notin A}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>fuzzy</td>
<td>$\forall x \in X: \mu_{\overline{A}}(x) = \sim \mu_A(x)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Intersection</th>
<th>classical</th>
<th>$A \cap B = {x \in X \mid x \in A \land x \in B}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>fuzzy</td>
<td>$\forall x \in X: \mu_{A \cap B}(x) = \top(\mu_A(x), \mu_B(x))$</td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Union</th>
<th>classical</th>
<th>$A \cup B = {x \in X \mid x \in A \lor x \in B}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>fuzzy</td>
<td>$\forall x \in X: \mu_{A \cup B}(x) = \bot(\mu_A(x), \mu_B(x))$</td>
<td></td>
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</tbody>
</table>
The fuzzy intersection shown on the left and the fuzzy union on the right are independent of the underlying $t$-norm and $t$-conorm, respectively.
- To describe a domain with linguistic terms it is fuzzy-partitioned with a collection of fuzzy sets.
  
  Every fuzzy set of the partition gets assigned a linguistic term.

- Usual condition: at every point of the domain the membership degrees must sum up to 1 (partition of unity).

**Example:** Fuzzy partition for temperatures

We define a linguistic variable with values *cold, tepid, warm* and *hot*.
• The knowledge base contains the fuzzy rules for the controller and the fuzzy partitions of the variables’ domains.

• A fuzzy rule reads: **if** $X_1$ **is** $A_{i_1}^{(1)}$ **and** ... **and** $X_n$ **is** $A_{i_n}^{(n)}$ **then** $Y$ **is** $B$.

  $X_1, \ldots, X_n$ are measured values and $Y$ is the control variable. $A_{i_k}^{(k)}$ and $B$ are linguistic terms with assigned fuzzy sets.
Abbreviations

pb – positive big
pm – positive medium
ps – positive small
az – approximately zero
ns – negative small
nm – negative medium
nb – negative big
Rule evaluation of a Mamdani–Assilian controller. The input tuple \((25, -4)\) leads to the fuzzy output shown on the right. The output value is determined from this fuzzy set by defuzzification, e.g. through the Mean-of-Maxima method (MOM) or the Center-of-Gravity method (COG).
A fuzzy rule system with one input and one output variable and three fuzzy rules. Every pyramid is defined by a fuzzy rule. The input value $x$ leads to the fuzzy output shaded in gray.
Defuzzification

The evaluation of the fuzzy rules results in an **output fuzzy set**.

This output fuzzy set has to be transformed into a **crisp control value**.

This task is called **defuzzification**.

The most important defuzzification methods are:

- **Center of Gravity (COG)**
  The center of gravity of the area under the output fuzzy set.

- **Center of Area (COA)**
  The point that divides the area under the output fuzzy set into equally sized parts.

- **Mean of Maxima (MOM)**
  The arithmetic mean of the locations with maximal membership degree.