Time Series Analysis

Motivation

Decomposition Models

- Additive models, multiplicative models

Global Approaches

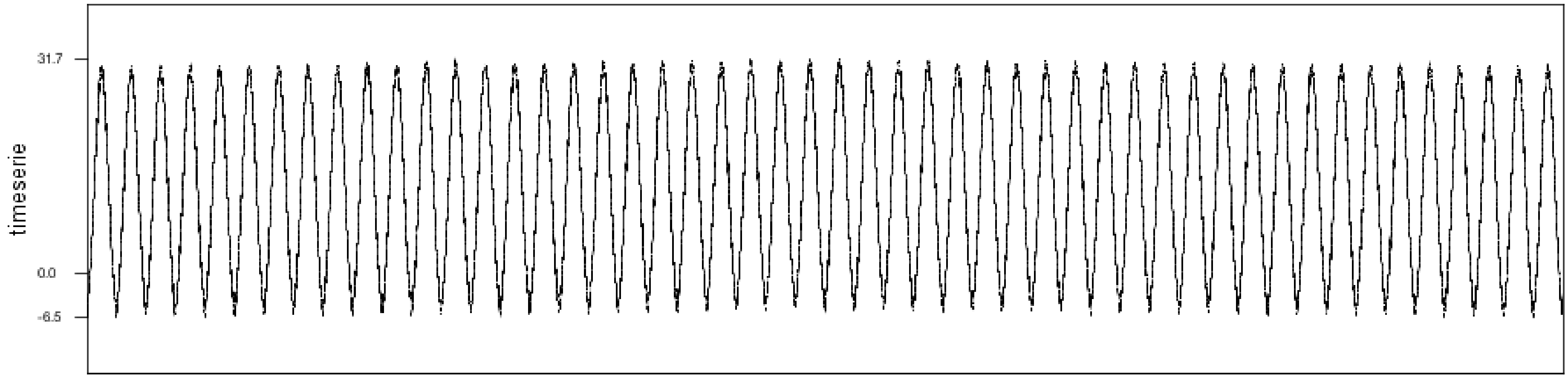
- Regression
- With and without seasonal component

Local Approaches

- Moving Averages Smoothing
- With and without seasonal component

Summary

Example: Temperatures data set (fictive)



The plot shows the average temperature per day for 50 years.

Is there any trend visible?

How to extract seasonal effects?

Decomposition Models

The time series is given as a sequence of values

$$y_1, \dots, y_t, \dots, y_n$$

We assume that every y_t is a composition of (some of) the following components:

- g_t trend component
- s_t seasonal component
- c_t cyclical variation
- ϵ_t irregular component (random factors, noise)

Assume a functional dependency:

$$y_t = f(g_t, s_t, c_t, \epsilon_t)$$

Components of Time Series

Trend Component

Reflects long-term developments.

Often assumed to be a monotone function of time.

Represents the actual component we are interested in.

Cyclic Component

Reflects mid-term developments.

Models economical cycles such as booms and recessions.

Variable cycle length.

We do not consider this component here.

Remark: Often, both components are combined.

Components of Time Series

Seasonal Component

Reflects short-term developments.

Constant cycle length (i. e., 12 months)

Represents changes that (re)occur rather regularly.

Irregular Component

Represents everything else that cannot be related to the other components.

Combines irregular changes, random noise and local fluctuations.

We assume that the values are small and have an average of zero.

Additive Decomposition

$$y_t = g_t + s_t + \epsilon_t$$

Pure trend model: $y_t = g_t + \epsilon_t$ (stock market, no season)

Possible extension: $y_t = g_t + s_t + x_t\beta + \epsilon_t$ (calendar effects)

Multiplicative Decomposition

$$y_t = g_t \cdot s_t \cdot \epsilon_t$$

Seasonal changes may increase with trend.

Transform into additive model:

$$\tilde{y}_t = \log y_t + \log s_t + \log \epsilon_t$$

Goal: Estimate the components from a given time series, i. e.

$$\hat{g}_t + \hat{s}_t + \epsilon_t \;\approx\; y_t$$

Application: With the estimates, we can compute the

trend-adjusted series: $y_t - \hat{g}_t$

season-adjusted series: $y_t - \hat{s}_t$

We only consider additive models here.

$\Rightarrow$ Additional assumptions necessary in order to find ways to infer the desired components.

Overview

Global approach: There is a fix functional dependence throughout the entire time range. (⇒ regression models)

Local approach: We do not postulate a global model and rather use local estimations to describe the respective components.

Seasonal effects: We have to decide beforehand whether to assume a seasonal component or not.

	Global	Local
without Season	Regression	Smoothing Averages
with Season	Dummy Variables	Smoothing Averages

Global Approach (without Season)

Model: $y_t = g_t + \epsilon_t$

Assumptions:

No seasonal component: $s_t = 0$

Depending on g_t, use regression analysis to estimate the parameter(s) to define the trend component.

- linear trend: $g_t = \beta_0 + \beta_1 t$
- quadratic trend: $g_t = \beta_0 + \beta_1 t + \beta_2 t^2$
- polynomial trend: $g_t = \beta_0 + \beta_1 t + \cdots + \beta_q t^q$
- exponential trend: $g_t = \beta_0 \exp(\beta_1 t)$
- logistic trend: $g_t = \frac{\beta_0}{\beta_1 + \exp(-\beta_2 t)}$

Global Approach (with Season)

Model: $y_t = s_t + \epsilon_t$ (no trend)

Assumptions:

No trend component: $g_t = 0$

Seasonal component does not change from period to period.

Introduce *dummy variables* for every time span (here: months) that serve as indicator functions to determine to which month a specific t belongs:

$$s_m(t) = \begin{cases} 1, & \text{if } t \text{ belongs to month } m \\ 0, & \text{otherwise} \end{cases}$$

The seasonal component is then set up as $s_t = \sum_{m=1}^{12} \beta_m s_m(t)$.

Determine the *monthly effects* β_m with normal least squares method.

Global Approach (with Season)

Model: $y_t = g_t + s_t + \epsilon_t$

Assumptions:

Estimate $\hat{g}_t$ while temporarily ignoring s_t.

Estimate s_t from the trend-adjusted $\tilde{y}_t = y_t - \hat{g}_t$.

Model: $y_t = \alpha_1 t + \cdots + \alpha_q t^q + \cdots + \beta_1 s_1(t) + \cdots + \beta_{12} s_{12}(t) + \epsilon_t$

Assumptions:

Seasonal component does not change from period to period.

Model the seasonal effects with trigonometric functions:

$$s_t = \beta_0 + \sum_{m=1}^{6} \beta_m \cos\left(2\pi \frac{m}{12} t\right) + \sum_{m=1}^{5} \gamma_m \sin\left(2\pi \frac{m}{12} t\right)$$

Determine $\alpha_1, \ldots, \alpha_q, \beta_0, \ldots, \beta_6$ and $\gamma_1, \ldots, \gamma_5$ with normal least squares method.

Local Approach (without Season)

General Idea: Smooth the time series.

Estimate the trend component g_t at time t as the average of the values around time t.

For a given time series $y_1, \ldots, y_n$, the **Smoothing Average** $y_t^\star$ of order r is defined as follows:

$$y_t^\star = \begin{cases} \dfrac{1}{2k+1} \cdot \displaystyle\sum_{j=-k}^{k} y_{t+j}, & \text{if } r = 2k+1 \\[2ex] \dfrac{1}{2k} \cdot \left(\dfrac{1}{2} y_{t-k} + \displaystyle\sum_{j=-k+1}^{k-1} y_{t+j} + \dfrac{1}{2} y_{t+k}\right), & \text{if } r = 2k \end{cases}$$

Local Approach (without Season)

Model: $y_t = g_t + \epsilon_t$

Assumptions:

In every time frame of width $2k + 1$ the time series can be assumed to be linear.

ϵ_t averages to zero.

Then we use the smoothing average to estimate the trend component:

$$\hat{g}_t = y_t^\star$$

Local Approach (with Season)

Model: $y_t = g_t + s_t + \epsilon_t$

Assumptions:

Seasonal component has period length p (repeats after p points):

$$s_t = s_{t+p}, \quad t = 1, \ldots, n - p$$

Sum of seasonal values is zero: $\sum_{j=1}^{p} s_j = 0$

Trend component is linear in time frames of width p (if p is odd) or $p + 1$ (if p is even).

Irregular component averages to zero.

Local Approach (with Season)

Let $k = \frac{p-1}{2}$ (for odd p) or $k = \frac{p}{2}$ (for even p).

Then:

Estimate the trend component with smoothing average:

$$\hat{g}_t = y_t^\star, \qquad k+1 \leq t \leq n-k$$

Estimate the seasonal components $s_1, \dots, s_p$ as follows:

$$\hat{s}_i = \tilde{s}_i - \frac{1}{p}\sum_{j=1}^{p} \tilde{s}_j \quad \text{with} \quad \tilde{s}_t \frac{1}{m_i - l_i + 1} \sum_{j=l}^{m_i} (y_{i+jp} - y_{i+jp}^\star), \qquad 1 \leq i \leq p$$

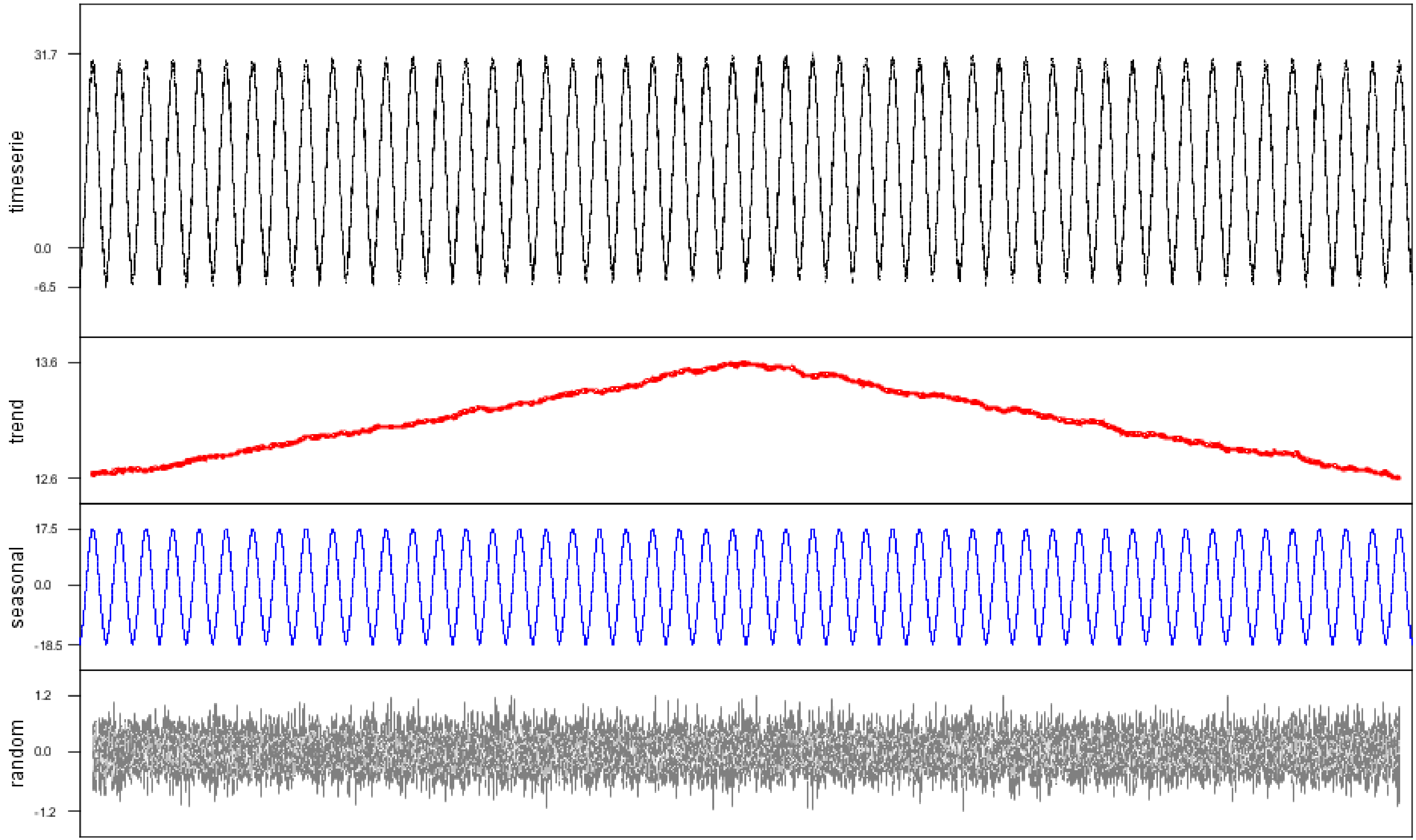
where

$$m_i = \max\{m \in \mathbb{N}_0 \mid i + mp \leq n - k\}$$

and

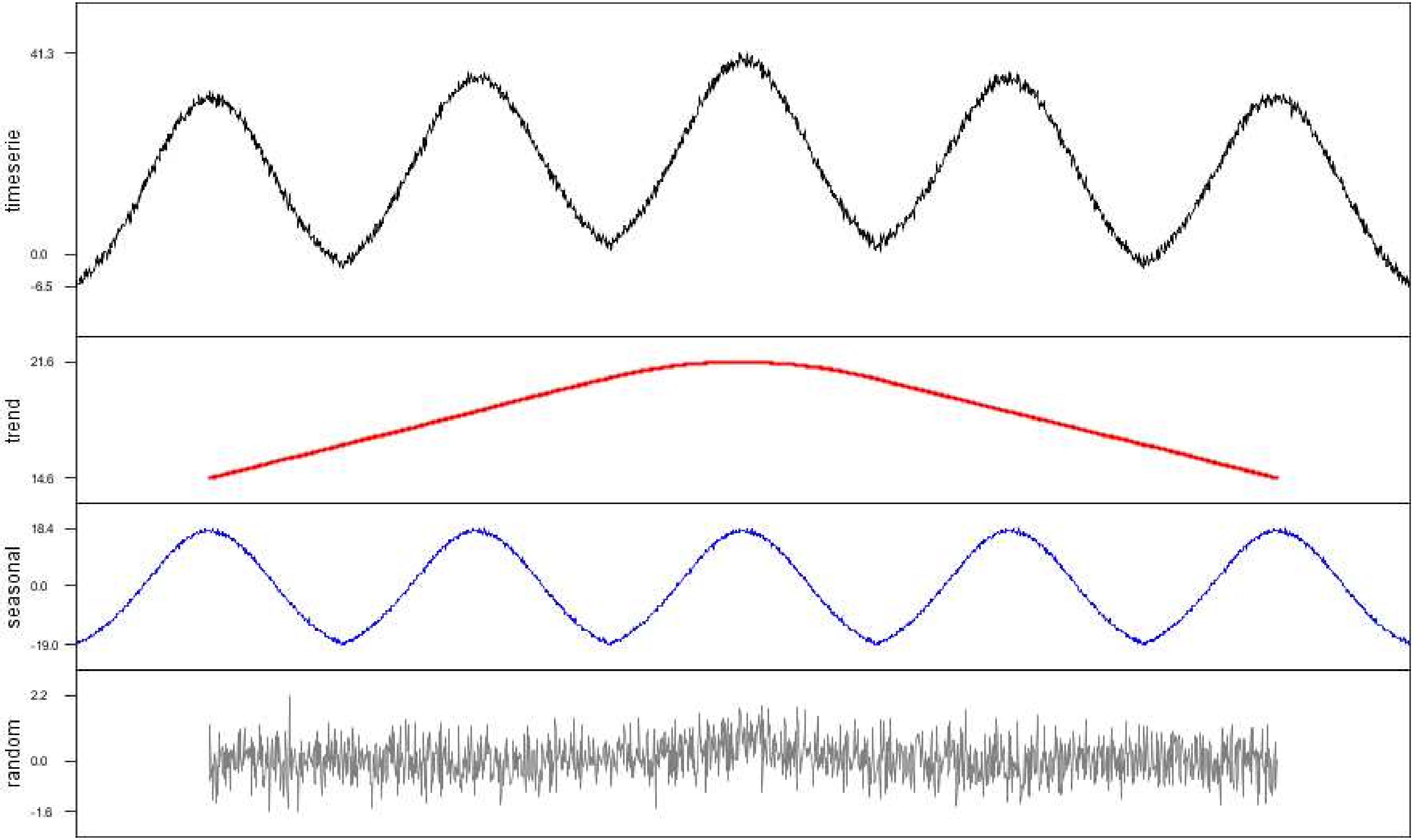
$$l_i = \min\{l \in \mathbb{N}_0 \mid i + lp \geq k + 1\}$$

Example (from motivation)



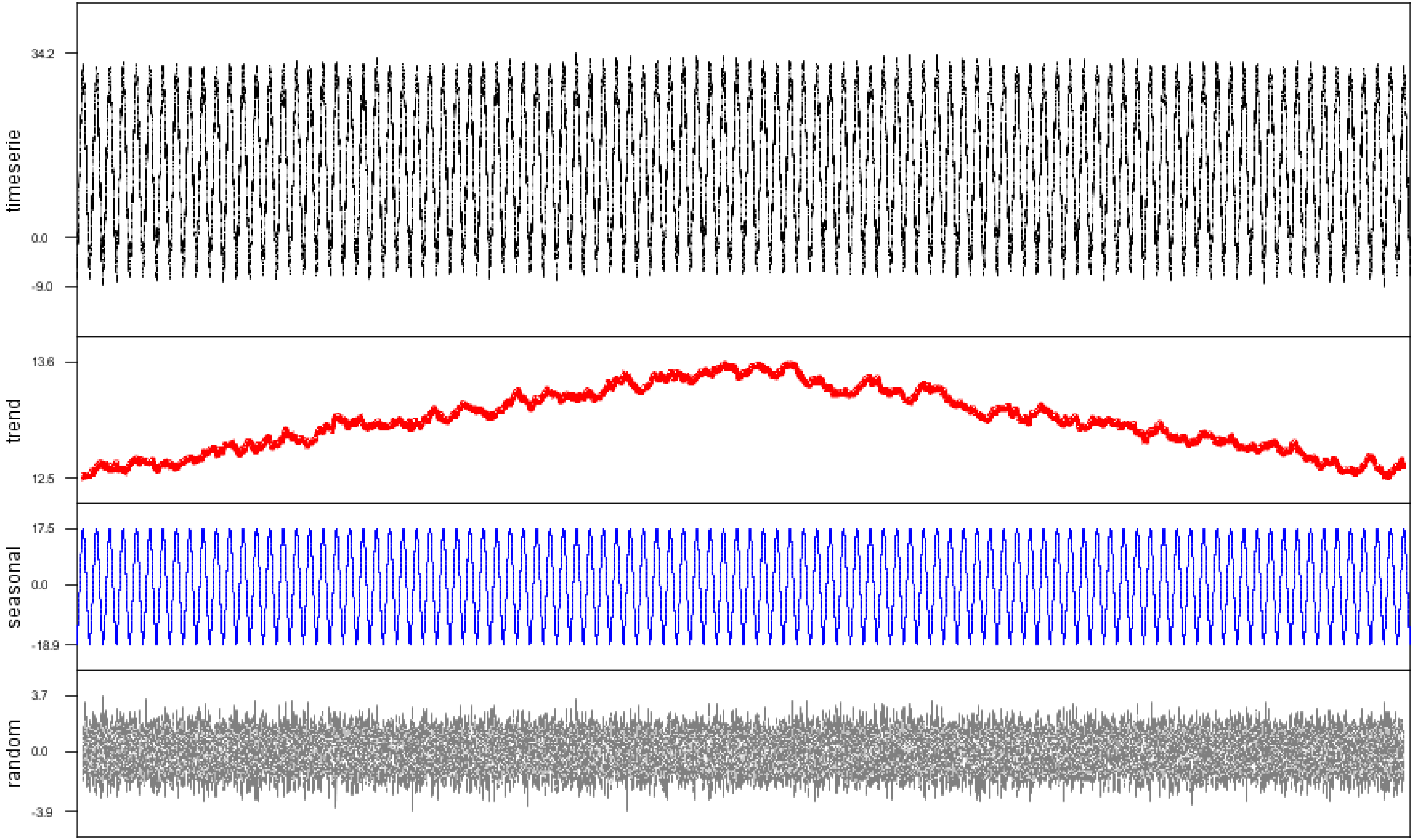
We can extract an increase and decrease of 1 degree during 50 years even though the amount of noise is more than twice as large than the actual trend.

Example



5 years period, trend ±8 degrees, noise amount ±2 degrees

Example



100 years period, trend ±1 degree, noise amount ±3 degrees

Summary

Definition of the problem domain

- Consider a time series to be composed of subcomponents.
- Additive and multiplicative models.

Global and local approaches

- With and without seasonal components.

Robust to noise

- Noise can be higher than the trend component itself.