Fuzzy Systems

Fuzzy Set Theory

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Outline

Definition of a "set"

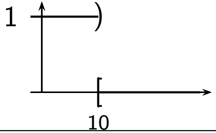
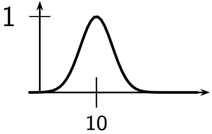
"By a set we understand every collection made into a whole of definite, distinct objects of our intuition or of our thought." (Georg Cantor).

For a set in Cantor's sense, the following properties hold:

- $x \neq \{x\}$.
- If $x \in X$ and $X \in Y$, then $x \notin Y$.
- The Set of all subsets of X is denoted as 2^X.
- $\emptyset$ is the empty set and thus very important.

Georg Cantor (1845-1918)

Extension to a fuzzy set

ling. description		model	
all numbers smaller than 10	*objective*		characteristic function of a set
all numbers <u>almost</u> equal to 10	*subjective*		membership function of a "fuzzy set"

Definition

A fuzzy set μ of $X \neq \emptyset$ is a function from the reference set X to the unit interval, *i.e.* $\mu : X \to [0,1]$. $\mathcal{F}(X)$ represents the set of all fuzzy sets of X, *i.e.* $\mathcal{F}(X) \stackrel{\text{def}}{=} \{\mu \mid \mu : X \to [0,1]\}$.

Vertical Representation

So far, fuzzy sets were described by
their characteristic/membership function and
assigning degree of membership $\mu(x)$ to each element $x \in X$.

That is the **vertical representation** of the corresponding fuzzy set, *e.g.* linguistic expression like "about m"

$$\mu_{m,d}(x) = \begin{cases} 1 - \left|\frac{m-x}{d}\right|, & \text{if } m - d \leq x \leq m + d \\ 0, & \text{otherwise,} \end{cases}$$

or "approximately between b and c"

$$\mu_{a,b,c,d}(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \leq x < b \\ 1, & \text{if } b \leq x \leq c \\ \frac{x-d}{c-d}, & \text{if } c < x \leq d \\ 0, & \text{if } x < a \text{ or } x > d. \end{cases}$$

Horizontal Representation

Another representation is very often applied as follows:

For all membership degrees α belonging to chosen subset of $[0, 1]$, human expert lists elements of X that fulfill vague concept of fuzzy set with degree $\geq \alpha$.

That is the **horizontal representation** of fuzzy sets by their α-**cuts**.

Definition
Let $\mu \in \mathcal{F}(X)$ and $\alpha \in [0, 1]$. Then the sets

$$[\mu]_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}, \quad [\mu]_{\underline{\alpha}} = \{x \in X \mid \mu(x) > \alpha\}$$

are called the α-*cut* and *strict* α-*cut* of μ.

A Simple Example

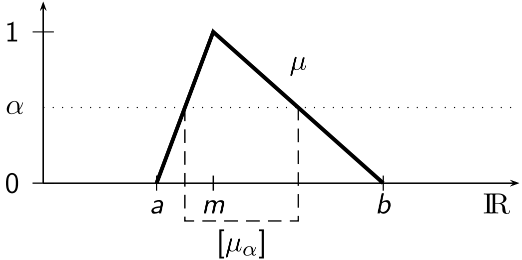
Let $A \subseteq X, \chi_A : X \to [0, 1]$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}$$

Then $[\chi_A]_\alpha = A$ for $0 < \alpha \leq 1$.

χ_A is called indicator function or characteristic function of A.

An Example



Let μ be triangular function on $\mathbb{R}$ as shown above.

α-cut of μ can be constructed by

1. drawing horizontal line parallel to x-axis through point $(0, \alpha)$,
2. projecting this section onto x-axis.

$$[\mu]_\alpha = \begin{cases} [a + \alpha(m - a), b - \alpha(b - m)], & \text{if } 0 < \alpha \leq 1, \\ \mathbb{R}, & \text{if } \alpha = 0. \end{cases}$$

Properties of α-cuts I

Any fuzzy set can be described by specifying its α-cuts.

That is the α-cuts are important for application of fuzzy sets.

Theorem

Let $\mu \in \mathcal{F}(X)$, $\alpha \in [0,1]$ *and* $\beta \in [0,1]$.

(a) $[\mu]_0 = X$,

(b) $\alpha < \beta \Longrightarrow [\mu]_\alpha \supseteq [\mu]_\beta$,

(c) $\bigcap_{\alpha:\alpha<\beta} [\mu]_\alpha = [\mu]_\beta$.

Properties of α-cuts II

Theorem (Representation Theorem)

Let $\mu \in \mathcal{F}(X)$. Then

$$\mu(x) = \sup_{\alpha \in [0,1]} \left\{ \min(\alpha, \chi_{[\mu]_\alpha}(x)) \right\}$$

where $\chi_{[\mu]_\alpha}(x) = \begin{cases} 1, & \text{if } x \in [\mu]_\alpha \\ 0, & \text{otherwise.} \end{cases}$

So, fuzzy set can be obtained as upper envelope of its α-cuts.

Simply draw α-cuts parallel to horizontal axis in height of α.

In applications it is recommended to select finite subset $L \subseteq [0, 1]$ of relevant degrees of membership.

They must be semantically distinguishable.

That is, fix level sets of fuzzy sets to characterize only for these levels.

System of Sets

In this manner we obtain **system of sets**

$$\mathcal{A} = (A_\alpha)_{\alpha \in L}, \quad L \subseteq [0,1], \quad \text{card}(L) \in \mathbb{N}.$$

$\mathcal{A}$ must satisfy consistency conditions for $\alpha, \beta \in L$:

(a) $0 \in L \Longrightarrow A_0 = X$, (fixing of reference set)

(b) $\alpha < \beta \Longrightarrow A_\alpha \supseteq A_\beta$. (monotonicity)

This induces fuzzy set

$$\mu_{\mathcal{A}} : X \to [0,1],$$
$$\mu_{\mathcal{A}}(x) = \sup_{\alpha \in L} \{\min(\alpha, \chi_{A_\alpha}(x))\}.$$

If L is not finite but comprises all values $[0,1]$, then μ must satisfy

(c) $\bigcap_{\alpha:\alpha<\beta} A_\alpha = A_\beta$. (condition for continuity)

Representation of Fuzzy Sets

Definition

$\mathcal{F}L(X)$ denotes the set of all families $(A_\alpha)_{\alpha\in[0,1]}$ of sets that satisfy

(a) $A_0 = X$,

(b) $\alpha < \beta \Longrightarrow A_\alpha \supseteq A_\beta$,

(c) $\bigcap\limits_{\alpha:\alpha<\beta} A_\alpha = A_\beta$.

Any family $\mathcal{A} = (A_\alpha)_{\alpha\in[0,1]}$ of sets of X that satisfy (a)–(b) represents fuzzy set $\mu_\mathcal{A} \in \mathcal{F}(X)$ with

$$\mu_\mathcal{A}(x) = \sup\left\{\alpha \in [0,1] \mid x \in A_\alpha\right\}.$$

Vice versa: If there is $\mu \in \mathcal{F}(X)$,
then family $([\mu]_\alpha)_{\alpha\in[0,1]}$ of α-cuts of μ satisfies (a)–(b).

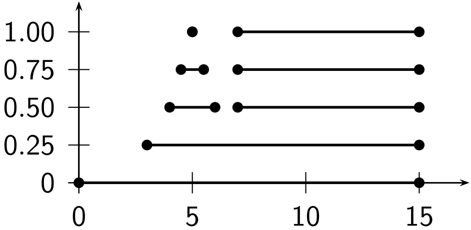
"Approximately 5 or greater than or equal to 7"

An Exemplary Horizontal View

Suppose that $X = [0, 15]$.

An expert chooses $L = \{0, 0.25, 0.5, 0.75, 1\}$ and α-cuts:

- $A_0 \quad = [0, 15]$,
- $A_{0.25} = [3, 15]$,
- $A_{0.5} \ = [4, 6] \cup [7, 15]$,
- $A_{0.75} = [4.5, 5.5] \cup [7, 15]$,
- $A_1 \quad = \{5\} \cup [7, 15]$.



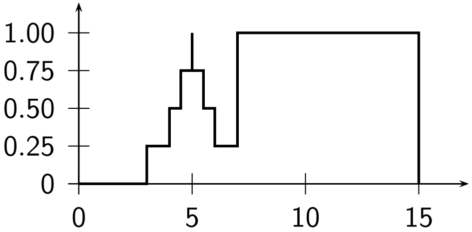
The family $(A_\alpha)_{\alpha \in L}$ of sets induces upper shown fuzzy set.

"Approximately 5 or greater than or equal to 7"

An Exemplary Vertical View

$\mu_{\mathcal{A}}$ is obtained as upper envelope of the family $\mathcal{A}$ of sets.

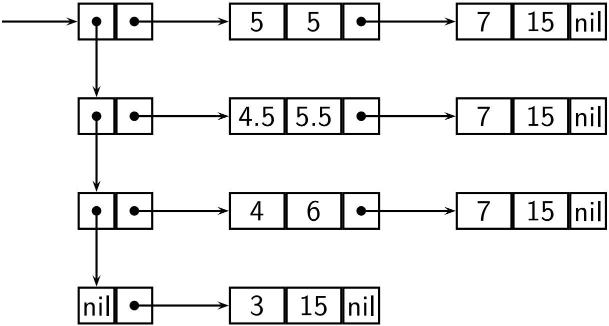
The difference between horizontal and vertical view is obvious:



The horizontal representation is easier to process in computers.

Also, restricting the domain of x-axis to a discrete set is usually done.

Horizontal Representation in the Computer



Fuzzy sets are usually stored as chain of linear lists.

For each α-level, $\alpha \neq 0$.

A finite union of closed intervals is stored by their bounds.

This data structure is appropriate for arithmetic operators.

Support and Core of a Fuzzy Set

Definition

The *support* $S(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the crisp set that contains all elements of X that have nonzero membership. Formally

$$S(\mu) = [\mu]_{\underline{0}} = \{x \in X \mid \mu(x) > 0\}\,.$$

Definition

The *core* $C(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the crisp set that contains all elements of X that have membership of one. Formally,

$$C(\mu) = [\mu]_1 = \{x \in X \mid \mu(x) = 1\}\,.$$

Height of a Fuzzy Set

Definition

The *height* $h(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the largest membership grade obtained by any element in that set. Formally,

$$h(\mu) = \sup_{x \in X} \{\mu(x)\}.$$

$h(\mu)$ may also be viewed as supremum of α for which $[\mu]_\alpha \neq \emptyset$.

Definition

A fuzzy set μ is called *normal, iff* $h(\mu) = 1$.
It is called *subnormal, iff* $h(\mu) < 1$.

Convex Fuzzy Sets I

Definition
Let X be a vector space. A fuzzy set $\mu \in \mathcal{F}(X)$ is called *fuzzy convex* if its α-cuts are convex for all $\alpha \in (0, 1]$.

The membership function of a convex fuzzy set **is not a** convex function.

The classical definition: The membership functions are actually **concave**.

Fuzzy Numbers

Definition

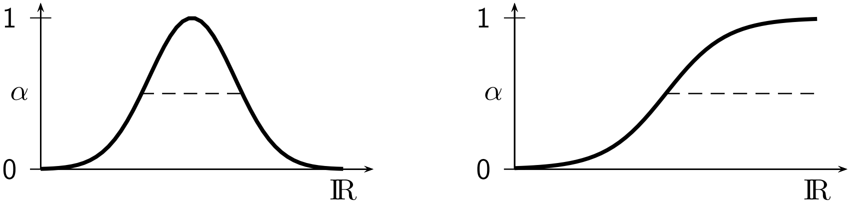
μ is a fuzzy number if and only if μ is normal and $[\mu]_\alpha$ is bounded, closed, and convex $\forall\alpha \in (0, 1]$.

Example:

The term *approximately* x_0 is often described by a parametrized class of membership functions, *e.g.*

$$\mu_1(x) = \max\{0,\ 1 - c_1|x - x_0|\}, \qquad c_1 > 0,$$
$$\mu_2(x) = \exp(-c_2\|x - x_0\|_p), \qquad c_2 > 0, \quad p \geq 1.$$

Convex Fuzzy Sets II



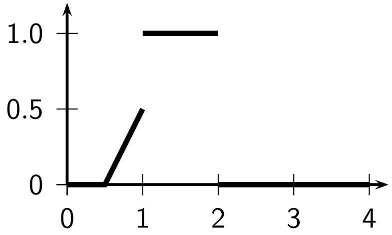
Theorem

A fuzzy set $\mu \in \mathcal{F}(\mathbb{R})$ *is convex if and only if*

$$\mu(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu(x_1), \mu(x_2)\}$$

for all $x_1, x_2 \in \mathbb{R}$ *and all* $\lambda \in [0, 1]$.

Fuzzy Numbers – Example



$$[\mu]_\alpha = \begin{cases} [1,2] & \text{if } \alpha \geq 0.5, \\ [0.5+\alpha, 2) & \text{if } 0 < \alpha < 0.5, \\ \mathbb{R} & \text{if } \alpha = 0 \end{cases}$$

Upper semi-continuous functions are often convenient in applications.

In many applications (*e.g.* fuzzy control) the class of the functions and their exact parameters have a limited influence on the results.

Only local monotonicity of the functions is really necessary.

In other applications (*e.g.* medical diagnosis) more precise membership degrees are needed.

Set Operators. . .

. . . are defined by using traditional logics operator

Let X be universe of discourse (universal set):

$$A \cap B = \{x \in X \mid x \in A \wedge x \in B\}$$
$$A \cup B = \{x \in X \mid x \in A \vee x \in B\}$$
$$A^c = \{x \in X \mid x \notin A\} = \{x \in X \mid \neg(x \in A)\}$$

$$A \subseteq B \text{ if and only if } (x \in A) \rightarrow (x \in B) \text{ for all } x \in X$$

One idea to define fuzzy set operators: use fuzzy logics.

The Traditional or Aristotlelian Logic

What is logic about? Different schools speak different languages!

There are raditional, linguistic, psychological, epistemological and mathematical schools.

Traditional logic has been founded by Aristotle (384-322 B.C.).

Aristotlelian logic can be seen as formal approach to human reasoning.

It's still used today in Artificial Intelligence for knowledge representation and reasoning about knowledge.

Detail of "The School of Athens" by R. Sanzio (1509) showing Plato (left) and his student Aristotle (right).

Outline

1. Representation of Fuzzy Sets

2. **Fuzzy Set Operators based on Multi-valued Logics**

   Boolean Algebra

   n-valued Logics

   Fuzzy Logic

Classical Logic: An Overview

Logic studies methods/principles of **reasoning**.

Classical logic deals with **propositions** (either *true* or *false*).

The *propositional logic* handles combination of *logical variables*.

Key idea: how to express n-ary logic functions with **logic primitives**, *e.g.* $\neg, \wedge, \vee, \rightarrow$.

A set of logic primitives is **complete** if any logic function can be composed by a finite number of these primitives,
e.g. $\{\neg, \wedge, \vee\}$, $\{\neg, \wedge\}$, $\{\neg, \rightarrow\}$, $\{\downarrow\}$ (NOR), $\{|\}$ (NAND)
(this was also discussed during the 1st exercise).

Inference Rules

When a variable represented by logical formula is:
true for all possible truth values, *i.e.* it is called **tautology**,
false for all possible truth values, *i.e.* it is called **contradiction**.

Various forms of tautologies exist to perform **deductive inference**

They are called **inference rules**:

$$(a \wedge (a \rightarrow b)) \rightarrow b \qquad \textit{(modus ponens)}$$

$$(\neg b \wedge (a \rightarrow b)) \rightarrow \neg a \qquad \textit{(modus tollens)}$$

$$((a \rightarrow b) \wedge (b \rightarrow c)) \rightarrow (a \rightarrow c) \qquad \textit{(hypothetical syllogism)}$$

e.g. modus ponens: given two true propositions a and $a \rightarrow b$ (*premises*), truth of proposition b (*conclusion*) can be inferred.

Every tautology remains a tautology when any of its variables is replaced with an arbitrary logic formula.

Boolean Algebra

The propositional logic based on finite set of logic variables is isomorphic to **finite set theory**.

Both of these systems are isomorphic to a finite **Boolean algebra**.

Definition

A *Boolean algebra* on a set B is defined as quadruple $\mathcal{B} = (B, +, \cdot, ^{-})$ where B has at least two elements (bounds) 0 and 1, $+$ and $\cdot$ are binary operators on B, and $^{-}$ is a unary operator on B for which the following properties hold.

Properties of Boolean Algebras I

(B1)	Idempotence	$a + a = a$	$a \cdot a = a$
(B2)	Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
(B3)	Associativity	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
(B4)	Absorption	$a + (a \cdot b) = a$	$a \cdot (a + b) = a$
(B5)	Distributivity	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	$a + (b \cdot c) = (a + b) \cdot (a + c)$
(B6)	Universal Bounds	$a + 0 = a,\ a + 1 = 1$	$a \cdot 1 = a,\ a \cdot 0 = 0$
(B7)	Complementary	$a + \overline{a} = 1$	$a \cdot \overline{a} = 0$
(B8)	Involution	$\overline{\overline{a}} = a$	
(B9)	Dualization	$\overline{a + b} = \overline{a} \cdot \overline{b}$	$\overline{a \cdot b} = \overline{a} + \overline{b}$

Properties (B1)-(B4) are common to every **lattice**,

i.e. a Boolean algebra is a distributive (B5), bounded (B6), and complemented (B7)-(B9) lattice,

i.e. every Boolean algebra can be characterized by a partial ordering on a set, *i.e.* $a \leq b$ if $a \cdot b = a$ or, alternatively, if $a + b = b$.

Set Theory, Boolean Algebra, Propositional Logic

Every theorem in one theory has a counterpart in each other theory.

Counterparts can be obtained applying the following substitutions:

Meaning	Set Theory	Boolean Algebra	Prop. Logic
values	2^X	B	$\mathcal{L}(V)$
"meet"/"and"	$\cap$	$\cdot$	$\wedge$
"join"/"or"	$\cup$	$+$	$\vee$
"complement"/"not"	^{c}	$^{-}$	$\neg$
identity element	X	1	1
zero element	$\emptyset$	0	0
partial order	$\subseteq$	$\leq$	$\rightarrow$

power set 2^X, set of logic variables V, set of all combinations $\mathcal{L}(V)$ of truth values of V

The Basic Principle of Classical Logic

The Principle of Bivalence:

“Every proposition is either true or false.”

It has been formally developed by Tarski.

Alfred Tarski (1902-1983)

Łukasiewicz suggested to replace it by
The Principle of Valence:

“Every proposition has a truth value.”

Propositions can have intermediate truth value,
expressed by a number from the unit interval $[0, 1]$.

Jan Łukasiewicz (1878-1956)

The Traditional or Aristotlelian Logic II

Short History

Aristotle introduced a logic of terms and drawing conclusion from two premises.

The great Greeks (Chrisippus) also developed logic of propositions.

Jan Łukasiewicz founded the multi-valued logic.

The multi-valued logic is to fuzzy set theory what classical logic is to set theory.

Three-valued Logics

A 2-valued logic can be extended to a 3-valued logic *in several ways*, *i.e.* different three-valued logics have been well established:

truth, falsity, indeterminacy are denoted by 1, 0, and 1/2, resp.

The negation $\neg a$ is defined as $1 - a$, *i.e.* $\neg 1 = 0, \neg 0 = 1$ and $\neg 1/2 = 1/2$.

Other primitives, *e.g.* $\wedge, \vee, \rightarrow, \leftrightarrow$, differ from logic to logic.

Five well-known three-valued logics (named after their originators) are defined in the following.

Primitives of Some Three-valued Logics

		Łukasiewicz				Bochvar				Kleene				Heyting				Reichenbach			
a	b	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
0	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$
0	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

All of them fully conform the usual definitions for $a, b \in \{0, 1\}$.

They differ from each other only in their treatment of $1/2$.

Question: Do they satisfy the law of contradiction $(a \wedge \neg a = 0)$ and the law of excluded middle $(a \vee \neg a = 1)$?

n-valued Logics

After the three-valued logics: generalizations to n-valued logics for arbitrary number of truth values $n \geq 2$.

In the 1930s, various n-valued logics were developed.

Usually truth values are assigned by rational number in $[0, 1]$.

Key idea: uniformly divide $[0, 1]$ into n truth values.

Definition

The set T_n of truth values of an n-valued logic is defined as

$$T_n = \left\{0 = \frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, \frac{n-1}{n-1} = 1\right\}.$$

These values can be interpreted as degree of truth.

Primitives in n-valued Logics

Łukasiewicz proposed first series of n-valued logics for $n \geq 2$.

In the early 1930s, he simply generalized his three-valued logic.

It uses truth values in T_n and defines primitives as follows:

$$\neg a = 1 - a$$

$$a \wedge b = \min(a, b) \qquad a \rightarrow b = \min(1,\ 1 + b - a)$$

$$a \vee b = \max(a, b) \qquad a \leftrightarrow b = 1 - |a - b|$$

The n-valued logic of Łukasiewicz is denoted by L_n.

The sequence $(L_2, L_3, \dots, L_\infty)$ contains the classical two-valued logic L_2 and an infinite-valued logic L_∞ (rational **countable** values T_∞).

The infinite-valued logic L_1 (**standard Łukasiewicz logic**) is the logic with all real numbers in $[0, 1]$ ($1 =$ cardinality of continuum).

Zadeh's fuzzy logic proposal was much simpler

In 1965, he proposed a logic with values in $[0, 1]$:

$$\begin{aligned} \neg a &= 1 - a, \\ a \wedge b &= \min(a, b), \\ a \vee b &= \max(a, b). \end{aligned}$$

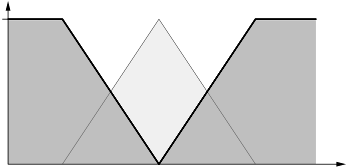
The set operators are defined pointwise as follows for μ, μ':

$$\begin{aligned} \neg\mu : X \to X, &\neg\mu(x) = 1 - \mu(x), \\ \mu \wedge \mu' : X \to X &(\mu \wedge \mu')(x) = \min\{\mu(x), \mu'(x)\}, \\ \mu \vee \mu' : X \to X &(\mu \vee \mu')(x) = \max\{\mu(x), \mu'(x)\}. \end{aligned}$$

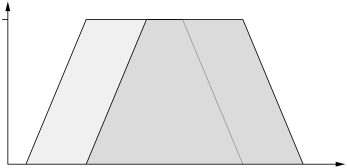
Zadeh in 2004

(born 1921)

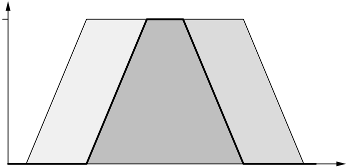
Standard Fuzzy Set Operators – Example



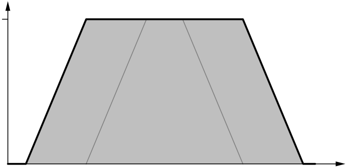
fuzzy complement



two fuzzy sets



fuzzy intersection



fuzzy union

Is Zadeh's logic a Boolean algebra?

Theorem
$(\mathcal{F}(X), \wedge, \vee, \neg)$ *is a complete distributive lattice but no Boolean algebra.*

Proof.
Consider $\mu : X \to X$ with $x \mapsto 0.5$, then $\neg\mu(x) = 0.5$ for all x and $\mu \wedge \neg\mu \neq \chi_{\emptyset}$. □