OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF

FAKULTÄT FÜR INFORMATIK

Fuzzy Systems

Fuzzy Rule Bases

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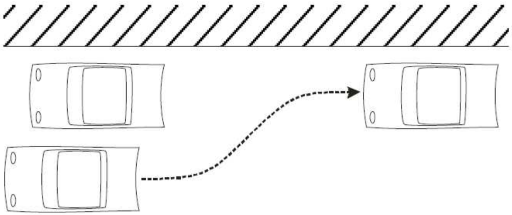
Outline

Example – Parking a car backwards



Questions:

What is the meaning of satisfactory parking?

Demand on precision?

Realization of control?

Fuzzy Control

Biggest success of fuzzy systems in industry and commerce.

Special kind of non-linear table-based control method.

Definition of non-linear transition function can be made without specifying each entry individually.

Examples: technical systems

- Electrical engine moving an elevator,
- Heating installation

Goal: define certain behavior

- Engine should maintain certain number of revolutions per minute.
- Heating should guarantee certain room temperature.

Table-based Control

Control systems all share a time-dependent **output variable**:

- Revolutions per minute,
- Room temperature.

Output is controlled by **control variable**:

- Adjustment of current,
- Thermostat.

Also, **disturbance variables** influence output:

- Load of elevator, . . . ,
- Outside temperature or sunshine through a window, . . .

Table-based Control

Computation of actual value incorporates both

control variable measurements of current output variable ξ and

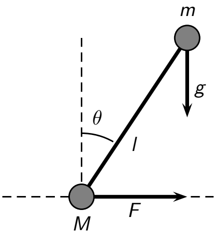
change of output variable $\Delta\xi = \frac{d\xi}{dt}$.

If ξ is given in finite time intervals,

then set $\Delta\xi(t_{n+1}) = \xi(t_{n+1}) - \xi(t_n)$.

In this case measurement of $\Delta\xi$ not necessary.

Example: Cartpole Problem



Balance an upright standing pole by moving its foot.

Lower end of pole can be moved unrestrained along horizontal axis.

Mass m at foot and mass M at head.

Influence of mass of shaft itself is negligible.

Determine force F (control variable) that is necessary to balance pole standing upright.

That is measurement of following output variables:

- angle θ of pole in relation to vertical axis,
- change of angle, *i.e.* triangular velocity $\dot{\theta} = \frac{d\theta}{dt}$.

Both should converge to zero.

Notation

Input variables $\xi_1, \ldots, \xi_n$, control variable η

Measurements: used to determine actual value of η

η may specify change of η.

Assumption: ξ_i, $1 \leq i \leq n$ is value of X_i, $\eta \in Y$

Solution: **control function** φ

$$\begin{aligned} \varphi : X_1 \times \ldots \times X_n &\to Y \\ (x_1, \ldots, x_n) &\mapsto y \end{aligned}$$

Example: Cartpole Problem (cont.)

Angle $\theta \in X_1 = [-90\,°, 90\,°]$

Theoretically, every angle velocity $\dot{\theta}$ possible.

Extreme $\dot{\theta}$ are artificially achievable.

Assume $-45\,°/\mathrm{s} \leq \dot{\theta} \leq 45\,°/\mathrm{s}$ holds,
i.e. $\dot{\theta} \in X_2 = [-45\,°/\mathrm{s}, 45\,°/\mathrm{s}]$.

Absolute value of force $|F| \leq 10\,\mathrm{N}$.

Thus define $F \in Y = [-10\,\mathrm{N}, 10\,\mathrm{N}]$.

Example: Cartpole Problem (cont.)

Differential equation of cartpole problem:

$$(M+m)\sin^2\theta\cdot l\cdot\ddot{\theta}+m\cdot l\cdot\sin\theta\cos\theta\cdot\dot{\theta}^2-(M+m)\cdot g\cdot\sin\theta=-F\cdot\cos\theta$$

Compute $F(t)$ such that $\theta(t)$ and $\dot{\theta}(t)$ converge towards zero quickly.

Physical analysis demands knowledge about physical process.

Problems of Classical Approach

Often very difficult or even impossible to specify accurate mathematical model.

Description with differential equations is very complex.

Profound physical knowledge from engineer.

Exact solution can be very difficult.

Should be possible: to control process without physical-mathematical model,
e.g. human being knows how to ride bike without knowing existence of differential equations.

Fuzzy Approach

Simulate behavior of human who knows how to control.

That is a **knowledge-based analysis**.

Directly ask expert to perform analysis.

Then expert specifies knowledge as **linguistic rules**, *e.g.* for cartpole problem:

"If θ is approximately zero and $\dot{\theta}$ is also approximately zero, then F has to be approximately zero, too."

Fuzzy Approach: Fuzzy Partitioning

1. Formulate set of linguistic rules:

Determine linguistic terms (represented by fuzzy sets).

$X_1, \ldots, X_n$ and Y is partitioned into fuzzy sets.

Define p_1 distinct fuzzy sets $\mu_1^{(1)}, \ldots, \mu_{p_1}^{(1)} \in \mathcal{F}(X_1)$ on set X_1.

Associate linguistic term with each set.

Fuzzy Approach: Fuzzy Partitioning II

Of set X_1 corresponds to interval $[a, b]$ of real line, then $\mu_1^{(1)}, \ldots, \mu_{p_1}^{(1)} \in \mathcal{F}(X_1)$ are triangular functions

$$\begin{aligned} \mu_{x_0,\varepsilon} : [a, b] &\to [0, 1] \\ x &\mapsto 1 - \min\{\varepsilon \cdot |x - x_0|,\ 1\}. \end{aligned}$$

If $a < x_1 < \ldots < x_{p_1} < b$, only $\mu_2^{(1)}, \ldots, \mu_{p_1-1}^{(1)}$ are triangular.

Boundaries are treated differently.

Fuzzy Approach: Fuzzy Partitioning III

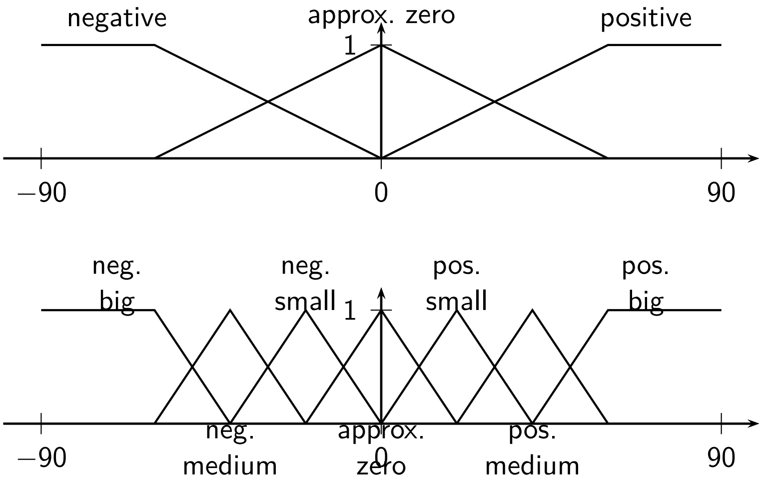
left fuzzy set:

$$\mu_1^{(1)} : [a,b] \to [0,1]$$
$$x \mapsto \begin{cases} 1, & \text{if } x \le x_1 \\ 1 - \min\{\varepsilon \cdot (x - x_1),\ 1\} & \text{otherwise} \end{cases}$$

right fuzzy set:

$$\mu_{p_1}^{(1)} : [a,b] \to [0,1]$$
$$x \mapsto \begin{cases} 1, & \text{if } x_{p_1} \le x \\ 1 - \min\{\varepsilon \cdot (x_{p_1} - x),\ 1\} & \text{otherwise} \end{cases}$$

Coarse and Fine Fuzzy Partitions



Example: Cartpole Problem (cont.)

X_1 partitioned into 7 fuzzy sets.

Support of fuzzy sets: intervals with length $\frac{1}{4}$ of whole range X_1.

Similar fuzzy partitions for X_2 and Y.

Next step: specify rules

$$\text{if } \xi_1 \text{ is } A^{(1)} \text{ and } \ldots \text{ and } \xi_n \text{ is } A^{(n)} \text{ then } \eta \text{ is } B,$$

$A^{(1)}, \ldots, A^{(n)}$ and B represent linguistic terms corresponding to $\mu^{(1)}, \ldots, \mu^{(n)}$ and μ according to $X_1, \ldots, X_n$ and Y.

Rule base consists of k rules.

Example: Cartpole Problem (cont.)

		θ						
		nb	nm	ns	az	ps	pm	pb
	nb			ps	pb			
	nm				pm			
	ns	nm		ns	ps			
$\dot{\theta}$	az	nb	nm	ns	az	ps	pm	pb
	ps				ns	ps		pm
	pm				nm			
	pb				nb	ns		

19 rules for cartpole problem, *e.g.*

If θ is *approximately zero* and $\dot{\theta}$ is *negative medium* then F is *positive medium*.

Fuzzy Approach: Challenge

How to define function $\varphi : X \to Y$ that fits to rule set?

Idea:

Represent set of rules as fuzzy relation.

Specify desired table-based controller by this fuzzy relation.

Fuzzy Relation

Consider only crisp sets.

Then, solving control problem = specifying control function $\varphi : X \to Y$.

φ corresponds to relation

$$R_\varphi = \{(x, \varphi(x)) \mid x \in X\} \subseteq X \times Y.$$

For measured input $x \in X$, control value

$$\{\varphi(x)\} = \{x\} \circ R_\varphi.$$

Fuzzy Control Rules

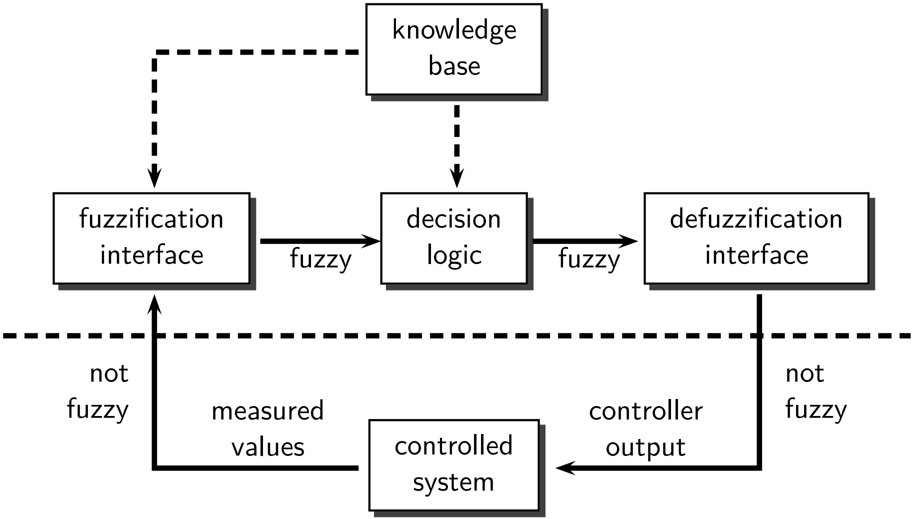
If temperature is very high **and** pressure is slightly low,

then heat change should be slightly negative.

If rate of descent = positive big **and** airspeed = negative big **and** glide slope = positive big,

then rpm change = positive big **and** elevator angle change = insignificant change.

Architecture of a Fuzzy Controller I



Architecture of a Fuzzy Controller II

Fuzzification interface

- receives current input value (eventually maps it to suitable domain),
- converts input value into linguistic term or into fuzzy set.

Knowledge base (consists of **data base** and **rule base**)

- Data base contains information about boundaries, possible domain transformations, and fuzzy sets with corresponding linguistic terms.
- Rule base contains linguistic control rules.

Decision logic (represents processing unit)

- computes output from measured input accord. to knowledge base.

Defuzzification interface (represents processing unit)

- determines crisp output value
  (and eventually maps it back to appropriate domain).

Outline

1. Fuzzy Control

2. **Fuzzy Rule Bases**

    Approximate Reasoning

    Disjunctive Rules

    Conjunctive Rules

    Fuzzy Relational Equations

Approximate Reasoning with Fuzzy Rules

General schema

Rule 1:	**if** X is M_1, **then** Y is N_1
Rule 2:	**if** X is M_2, **then** Y is N_2
$\vdots$	$\vdots$
Rule r:	**if** X is M_r, **then** Y is N_r
Fact:	X is M'
Conclusion:	Y is N'

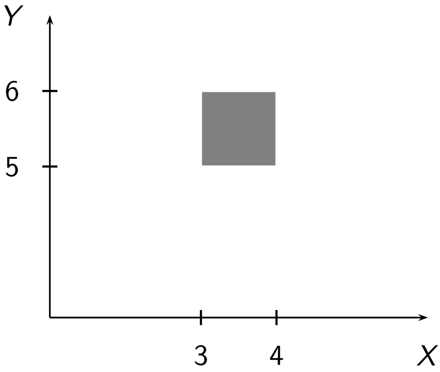
Given r **if-then rules** and fact “X is M'”, we conclude “Y is N'”.

Typically used in **fuzzy controllers**.

Approximate Reasoning
Disjunctive Imprecise Rule

Imprecise rule: **if** $X = [3, 4]$ **then** $Y = [5, 6]$.

Interpretation: values coming from $[3, 4] \times [5, 6]$.

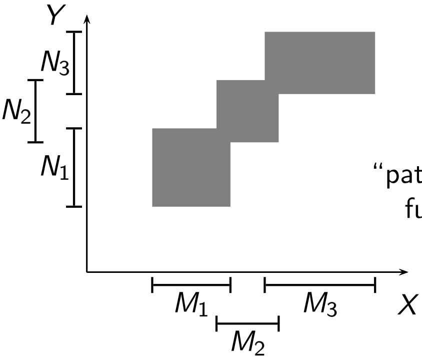


Approximate Reasoning

Disjunctive Imprecise Rules

Several imprecise rules: **if** $X = M_1$ **then** $Y = N_1$, **if** $X = M_2$ **then** $Y = N_2$, **if** $X = M_3$ **then** $Y = N_3$.

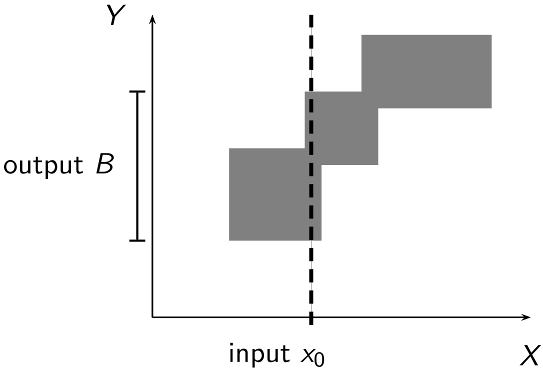
Interpretation: rule 1 as well as rule 2 as well as rule 3 hold true.



$$S = \bigcup_{i=1}^{r} M_i \times N_i$$

"patchwork rug" describing function's behavior as indicator function

Approximate Reasoning: Conclusion



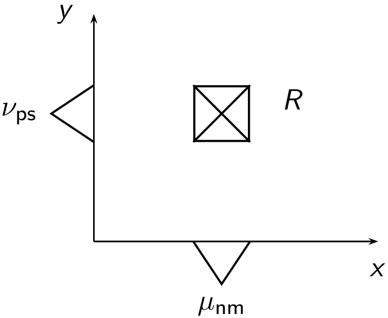
$$\{x_0\} \circ S = B$$

Approximate Reasoning

Disjunctive Fuzzy Rules

one fuzzy rule:

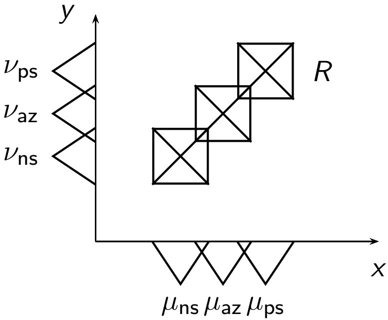
if $X = \text{nm}$ **then** $Y = \text{ps}$



$$R = \mu_{\text{nm}} \times \nu_{\text{ps}}$$

several fuzzy rules:

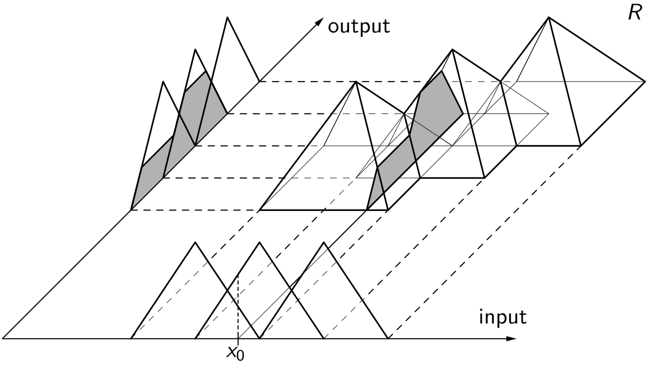
ns $\to$ ns', az $\to$ az', ps $\to$ ps'



$$R = \mu_{\text{ns}} \times \nu_{\text{ns'}} \cup \mu_{\text{az}} \times \nu_{\text{az'}} \cup \mu_{\text{ps}} \times \nu_{\text{ps'}}$$

Approximate Reasoning: Conclusion

Disjunctive Fuzzy Rules



3 fuzzy rules.

Every pyramid is specified by 1 fuzzy rule (Cartesian product).

Input x_0 leads to gray-shaded fuzzy output $\{x_0\} \circ R$.

Disjunctive or Conjunctive?

Fuzzy relation R employed in reasoning is obtained as follows.

For each rule i, we determine relation R_i by

$$R_i(x, y) = \min[M_i(x), N_i(y)]$$

for all $x \in X$, $y \in Y$.

Then, R is defined by union of R_i, *i.e.*

$$R = \bigcup_{1 \leq i \leq r} R_i.$$

That is, if-then rules are treated **disjunctive**.

If-then rules can be also treated **conjunctive** by

$$R = \bigcap_{1 \leq i \leq r} R_i.$$

Disjunctive or Conjunctive?

Decision depends on intended use and how R_i are obtained.

For both interpretations, two possible ways of applying composition:

$$B'_1 = A' \circ \left(\bigcup_{1 \le i \le r} R_i \right) \qquad B'_2 = A' \circ \left(\bigcap_{1 \le i \le r} R_i \right)$$

$$B'_3 = \bigcup_{1 \le i \le r} A' \circ R_i \qquad B'_4 = \bigcap_{1 \le i \le r} A' \circ R_i$$

Theorem

$B'_2 \subseteq B'_4 \subseteq B'_1 = B'_3$

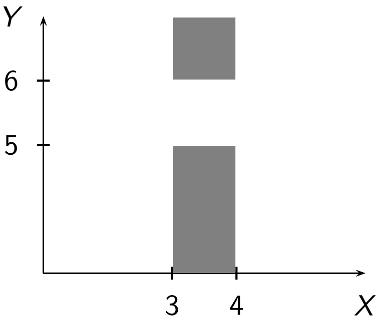
This holds for any continuous $\top$ used in composition.

Approximate Reasoning

Conjunctive Imprecise Rules

if $X = [3, 4]$ **then** $Y = [5, 6]$

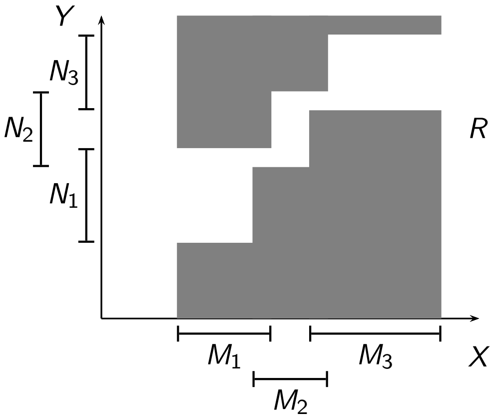
Gray-shaded values are impossible, white ones are possible.



Approximate Reasoning

Conjunctive Imprecise Rules

Several imprecise rules: **if** $X = M_1$ **then** $Y = N_1$, **if** $X = M_2$ **then** $Y = N_2$, **if** $X = M_3$ **then** $Y = N_3$.

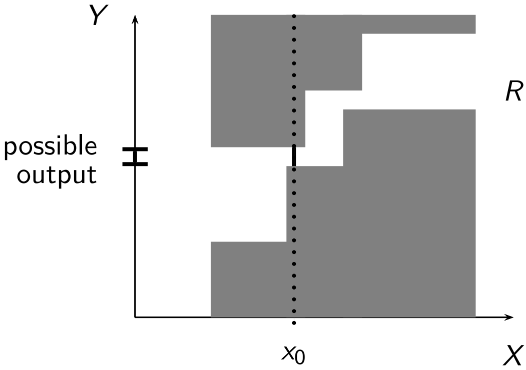


still possible are

$$R = \bigcap_{i=1}^{r} (M_i \times N_i) \cup (M_i^C \times Y)$$

"corridor" describing function's behavior

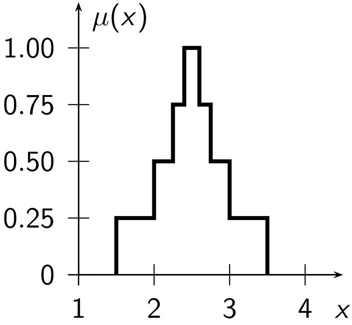
Approximate Reasoning with Crisp Input

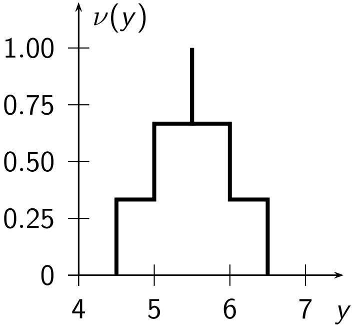


$$\text{output} = \{x_0\} \circ R$$

Generalization to Fuzzy Rules

if X is approx. 2.5 **then** Y is approx. 5.5

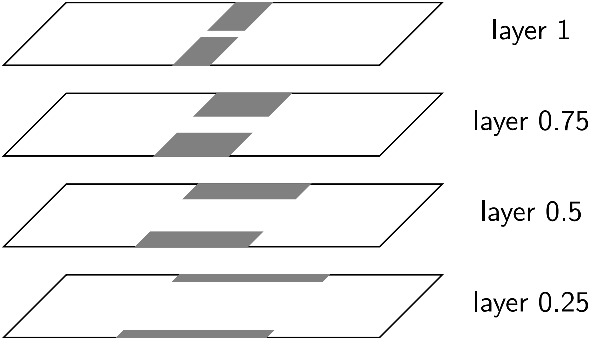




Modeling a Fuzzy Rule in Layers

Using Gödel Implication

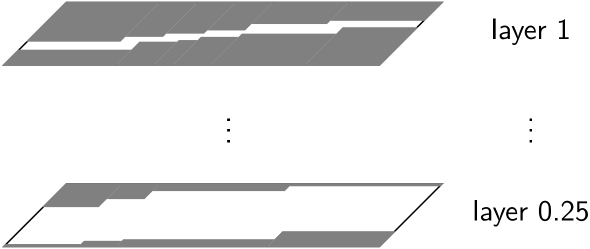
$$R_1 : \textbf{if } X = \mu_{M_1} \textbf{ then } Y = \nu_{B_1}$$



$$\mu_{R_1} : X \times Y \to [0,1], \quad I(x,y) = \begin{cases} 1 & \text{if } \mu_{M_1}(x) \leq \nu_{B_1}(y), \\ \nu_{B_1}(y) & \text{otherwise.} \end{cases}$$

Conjunctive Fuzzy Rule Base

R_1 : **if** $X = \mu_{M_1}$ **then** $Y = \nu_{B_1}$, $\quad \ldots, \quad$ R_n : **if** $X = \mu_{M_n}$ **then** $Y = \nu_{B_n}$



$$\mu_R = \min_{1 \le i \le r} \mu_{R_i}$$

Input μ_A, then output η with

$$\eta(y) = \sup_{x \in X} \min \{\mu_A(x), \mu_R(x, y)\}.$$

Example: Fuzzy Relation

Classes of cars $X = \{s, m, h\}$ (small, medium, high quality).

Possible maximum speeds $Y = \{140, 160, 180, 200, 220\}$ (in km/h).

Gor any $(x, y) \in X \times Y$, fuzzy relation ϱ states possibility that maximum speed of car of class x is y.

ϱ	140	160	180	200	220
s	1	.5	.1	0	0
m	0	.5	1	.5	0
h	0	0	.4	.8	1

Fuzzy Relational Equations

Given $\mu_1, \dots, \mu_r$ of X and $\nu_1, \dots, \nu_r$ of Y and r rules *if* μ_i *then* ν_i.

What is a **fuzzy relation** ϱ that fits the rule system?

One solution is to find a relation ϱ such that

$$\forall i \in \{1, \dots, r\} : \nu_i = \mu_i \circ \varrho,$$

$$\mu \circ \varrho : Y \to [0, 1], \quad y \mapsto \sup_{x \in X} \min\{\mu(x),\ \varrho(x, y)\}.$$

Solution of a Relational Equation

Theorem

i) *Let "if A then B" be a rule with* $\mu_A \in \mathcal{F}(X)$ *and* $\nu_B \in \mathcal{F}(Y)$*.*
Then the relational equation $\nu_B = \mu_A \circ \varrho$ *can be solved iff the Gödel relation* $\varrho_{A \circleddash B}$ *is a solution.*
$\varrho_{A \circleddash B} : X \times Y \to [0, 1]$ *is defined by*

$$(x, y) \mapsto \begin{cases} 1 & \text{if } \mu_A(x) \leq \nu_B(y), \\ \nu_B(y) & \text{otherwise.} \end{cases}$$

ii) *If* ϱ *is a solution, then the set of solutions* $R = \{\varrho_S \in \mathcal{F}(X \times Y) \mid \nu_B = \mu_A \circ \varrho_S\}$ *has the following property: If* $\varrho_{S'}, \varrho_{S''} \in R$*, then* $\varrho_{S' \cup S''} \in R$*.*

iii) *If* $\varrho_{A \circleddash B}$ *is a solution, then* $\varrho_{A \circleddash B}$ *is the largest solution w.r.t.* $\subseteq$*.*

Example

$$\mu_A = \begin{pmatrix} .9 & 1 & .7 \end{pmatrix} \qquad \nu_B = \begin{pmatrix} 1 & .4 & .8 & .7 \end{pmatrix}$$

$$\varrho_{A \circledsim B} = \begin{pmatrix} 1 & .4 & .8 & .7 \\ 1 & .4 & .8 & .7 \\ 1 & .4 & 1 & 1 \end{pmatrix} \qquad \varrho_1 = \begin{pmatrix} 0 & 0 & 0 & .7 \\ 1 & .4 & .8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccc|cccc} & & & 1 & .4 & .8 & .7 \\ & & & 1 & .4 & .8 & .7 \\ & & & 1 & .4 & 1 & 1 \\ \hline .9 & 1 & .7 & 1 & .4 & .8 & .7 \end{array} \qquad \varrho_2 = \begin{pmatrix} 0 & .4 & .8 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & .7 \end{pmatrix}$$

$\varrho_{A \circledsim B}$ largest solution, ϱ_1, ϱ_2 are two minimal solutions.

Solution space forms upper semilattice.

Solution of a Set of Relational Equations

Generalization of this result to system of r relational equations:

Theorem

Let $\nu_{B_i} = \mu_{A_i} \circ \varrho$ for $i = 1, \dots, r$ be a system of relational equations.

i) There is a solution iff $\bigcap_{i=1}^{r} \varrho_{A_i \circledS B_i}$ is a solution.

ii) If $\bigcap_{i=1}^{r} \varrho_{A_i \circledS B_i}$ is a solution, then this solution is the biggest solution w.r.t. $\subseteq$.

Remark: if there is no solution, then Gödel relation is at least a good approximation.

Solving a System of Relational Equations

Sometimes it is a good choice not to use the largest but a smaller solution.

i.e. the **Cartesian product** $\varrho_{A\times B}(x, y) = \min\{\mu_A(x), \nu_B(y)\}$.

If a solution of the relational equation $\nu_B = \mu_A \circ \varrho$ for ϱ exists, then $\varrho_{A\times B}$ is a solution, too.

Theorem

Let $\mu_A \in \mathcal{F}(X)$, $\nu_B \in \mathcal{F}(Y)$. Furthermore, let $\varrho \in \mathcal{F}(X \times Y)$ be a fuzzy relation which satisfies the relational equation $\nu_B = \mu_A \circ \varrho$. Then $\nu_B = \mu_A \circ \varrho_{A\times B}$ holds.

Solving a System of Relational Equations

Using Cartesian product

$\mu_{A_i} = \nu_{B_i} \circ \varrho,\ 1 \leq i \leq r$ can be reasonably solved with $A \times B$ by

$$\varrho = \max\left\{\varrho_{A_i \times B_i} \mid 1 \leq i \leq r\right\}.$$

For crisp value $x_0 \in X$ (represented by $\mathbb{1}_{\{x_0\}}$):

$$\begin{aligned}
\nu(y) &= \left(\mathbb{1}_{\{x_0\}} \circ \varrho\right)(y) \\
&= \max_{1 \leq i \leq r} \left\{ \sup_{x \in X} \min\{\mathbb{1}_{\{x_0\}}(x), \varrho_{A_i \times B_i}(x, y)\} \right\} \\
&= \max_{1 \leq i \leq r} \{\min\{\mu_{A_i}(x_0), \nu_{B_i}(y)\}.\}
\end{aligned}$$

That is Mamdani-Assilian fuzzy control (to be discussed).