

Fuzzy Systems

Fuzzy Logic

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Outline

1. Complement

Strict and Strong Negations

Families of Negations

Representation of Negations

2. Intersection and Union

3. Implication

Fuzzy Complement/Fuzzy Negation

Definition

Let X be a given set and $\mu \in \mathcal{F}(X)$. Then the *complement* $\bar{\mu}$ can be defined pointwise by $\bar{\mu}(x) := \sim(\mu(x))$ where $\sim : [0, 1] \rightarrow [0, 1]$ satisfies the conditions

$$\sim(0) = 1, \quad \sim(1) = 0$$

and

for $x, y \in [0, 1]$, $x \leq y \implies \sim x \geq \sim y$ (\sim is non-increasing).

Abbreviation: $\sim x := \sim(x)$

Strict and Strong Negations

Additional properties may be required

- $x, y \in [0, 1], x < y \implies \sim x > \sim y$ (\sim is strictly decreasing)
- \sim is continuous
- $\sim \sim x = x$ for all $x \in [0, 1]$ (\sim is involutive)

According to conditions, two subclasses of negations are defined:

Definition

A negation is called *strict* if it is also strictly decreasing and continuous. A strict negation is said to be *strong* if it is involutive, too.

$\sim x = 1 - x^2$, for instance, is strict, not strong, thus not involutive

Families of Negations

standard negation:

$$\sim x = 1 - x$$

threshold negation:

$$\sim_{\theta}(x) = \begin{cases} 1 & \text{if } x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

Cosine negation:

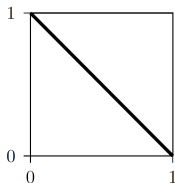
$$\sim x = \frac{1}{2}(1 + \cos(\pi x))$$

Sugeno negation:

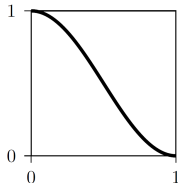
$$\sim_{\lambda}(x) = \frac{1-x}{1+\lambda x}, \quad \lambda > -1$$

Yager negation:

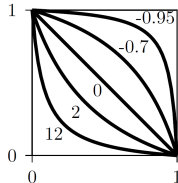
$$\sim_{\lambda}(x) = (1 - x^{\lambda})^{\frac{1}{\lambda}}$$



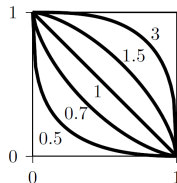
standard



cosine



Sugeno



Yager

Two Extreme Negations

$$\textit{intuitionistic negation } \sim_i(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

$$\textit{dual intuitionistic negation } \sim_{di}(x) = \begin{cases} 1 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

Both negations are not strictly increasing, not continuous, not involutive

Thus they are neither strict nor strong

They are “optimal” since their notions are nearest to crisp negation

\sim_i and \sim_{di} are two extreme cases of negations

For any negation \sim the following holds

$$\sim_i \leq \sim \leq \sim_{di}$$

Inverse of a Strict Negation

Any strict negation \sim is strictly decreasing and continuous.

Hence one can define its inverse \sim^{-1} .

\sim^{-1} is also strict but in general differs from \sim .

$\sim^{-1} = \sim$ if and only if \sim is involutive.

Every strict negation \sim has a unique value $0 < s_{\sim} < 1$ such that $\sim s_{\sim} = s_{\sim}$.

s_{\sim} is called *membership crossover point*.

$A(a) > s_{\sim}$ if and only if $A^c(a) < s_{\sim}$ where A^c is defined via \sim .

$\sim^{-1}(s_{\sim}) = s_{\sim}$ always holds as well.

Representation of Negations

Any strong negation can be obtained from standard negation.

Let $a, b \in \mathbb{R}$, $a \leq b$.

Let $\varphi : [a, b] \rightarrow [a, b]$ be continuous and strictly increasing.

φ is called *automorphism* of the interval $[a, b] \subset \mathbb{R}$.

Theorem

A function $\sim : [0, 1] \rightarrow [0, 1]$ is a strong negation if and only if there exists an automorphism φ of the unit interval such that for all $x \in [0, 1]$ the following holds

$$\sim_{\varphi}(x) = \varphi^{-1}(1 - \varphi(x)).$$

$\sim_{\varphi}(x) = \varphi^{-1}(1 - \varphi(x))$ is called φ -transform of the standard negation.

Outline

1. Complement

2. Intersection and Union

Triangular Norms and Conorms

De Morgan Triplet

Examples

The Special Role of Minimum and Maximum

Continuous Archimedean t-norms and t-conorms

Families of Operations

3. Implication

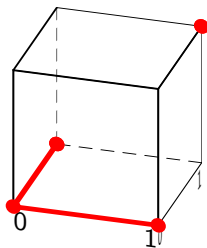
Classical Intersection and Union

Classical set intersection represents logical conjunction.

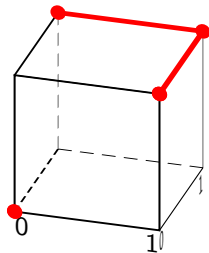
Classical set union represents logical disjunction.

Generalization from $\{0, 1\}$ to $[0, 1]$ as follows:

$x \wedge y$	0	1
0	0	0
1	0	1



$x \vee y$	0	1
0	0	1
1	1	1



Fuzzy Intersection and Union

Let A, B be fuzzy subsets of X , i.e. $A, B \in \mathcal{F}(X)$.

Their **intersection** and **union** can be defined pointwise using:

$$(A \cap B)(x) = \top(A(x), B(x)) \quad \text{where} \quad \top : [0, 1]^2 \rightarrow [0, 1]$$

$$(A \cup B)(x) = \perp(A(x), B(x)) \quad \text{where} \quad \perp : [0, 1]^2 \rightarrow [0, 1].$$

Triangular Norms and Conorms I

\top is a *triangular norm (t-norm)* $\iff \top$ satisfies conditions T1-T4

\perp is a *triangular conorm (t-conorm)* $\iff \perp$ satisfies C1-C4

for all $x, y \in [0, 1]$, the following laws hold

Identity Law

$$\mathbf{T1:} \quad \top(x, 1) = x \quad (A \cap X = A)$$

$$\mathbf{C1:} \quad \perp(x, 0) = x \quad (A \cup \emptyset = A).$$

Commutativity

$$\mathbf{T2:} \quad \top(x, y) = \top(y, x) \quad (A \cap B = B \cap A),$$

$$\mathbf{C2:} \quad \perp(x, y) = \perp(y, x) \quad (A \cup B = B \cup A).$$

Triangular Norms and Conorms II

for all $x, y, z \in [0, 1]$, the following laws hold

Associativity

$$\mathbf{T3:} \quad \top(x, \top(y, z)) = \top(\top(x, y), z) \quad (A \cap (B \cap C) = (A \cap B) \cap C),$$

$$\mathbf{C3:} \quad \perp(x, \perp(y, z)) = \perp(\perp(x, y), z) \quad (A \cup (B \cup C) = (A \cup B) \cup C).$$

Monotonicity

$y \leq z$ implies

$$\mathbf{T4:} \quad \top(x, y) \leq \top(x, z)$$

$$\mathbf{C4:} \quad \perp(x, y) \leq \perp(x, z).$$

Triangular Norms and Conorms III

\top is a *triangular norm (t-norm)* $\iff \top$ satisfies conditions T1-T4

\perp is a *triangular conorm (t-conorm)* $\iff \perp$ satisfies C1-C4

Both identity law and monotonicity respectively imply

$$\forall x \in [0, 1] : \top(0, x) = 0,$$

$$\forall x \in [0, 1] : \perp(1, x) = 1,$$

for any *t-norm* $\top : \top(x, y) \leq \min(x, y)$,

for any *t-conorm* $\perp : \perp(x, y) \geq \max(x, y)$.

note: $x = 1 \Rightarrow T(0, 1) = 0$ and

$x \leq 1 \Rightarrow T(x, 0) \leq T(1, 0) = T(0, 1) = 0$

De Morgan Triplet I

For every \top and strong negation \sim , one can define t -conorm \perp by

$$\perp(x, y) = \sim \top(\sim x, \sim y), \quad x, y \in [0, 1].$$

Additionally, in this case $\top(x, y) = \sim \perp(\sim x, \sim y)$, $x, y \in [0, 1]$.

\perp, \top are called *N-dual t-conorm* and *N-dual t-norm* to \top, \perp , resp.

In case of the standard negation $\sim x = 1 - x$ for $x \in [0, 1]$,
N-dual \perp and \top are called *dual t-conorm* and *dual t-norm*, resp.

$\perp(x, y) = \sim \top(\sim x, \sim y)$ expresses “fuzzy ” De Morgan’s law.

note: De Morgan’s laws $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

De Morgan Triplet II

Definition

The triplet (\top, \perp, \sim) is called *De Morgan triplet* if and only if

\top is *t*-norm, \perp is *t*-conorm, \sim is strong negation,

\top, \perp and \sim satisfy $\perp(x, y) = \sim \top(\sim x, \sim y)$.

In the following, some important De Morgan triplets will be shown, only the most frequently used and important ones.

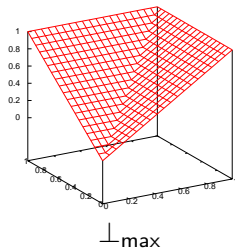
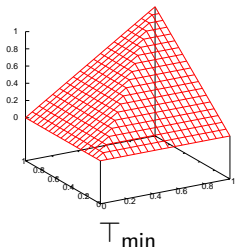
In all cases, the standard negation $\sim x = 1 - x$ is considered.

The Minimum and Maximum I

$$\top_{\min}(x, y) = \min(x, y), \quad \perp_{\max}(x, y) = \max(x, y)$$

Minimum is the greatest t -norm and max is the weakest t -conorm.

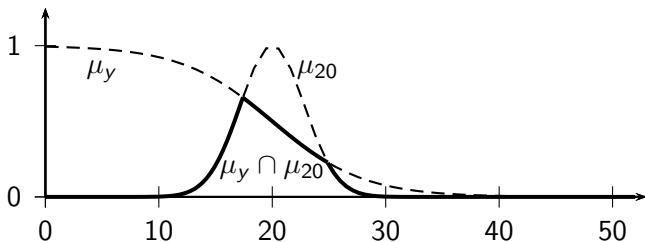
$\top(x, y) \leq \min(x, y)$ and $\perp(x, y) \geq \max(x, y)$ for any \top and \perp



The Minimum and Maximum II

\top_{\min} and \perp_{\max} can be easily processed numerically and visually, e.g. linguistic values *young* and *approx. 20* described by μ_y , μ_{20} .

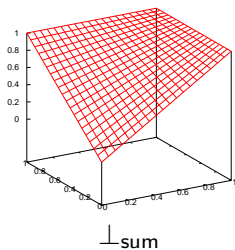
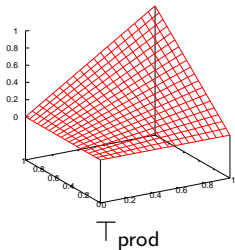
$\top_{\min}(\mu_y, \mu_{20})$ is shown below.



The Product and Probabilistic Sum

$$\top_{\text{prod}}(x, y) = x \cdot y, \quad \perp_{\text{sum}}(x, y) = x + y - x \cdot y$$

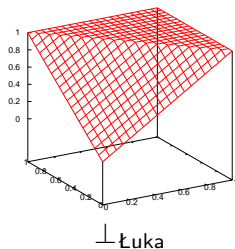
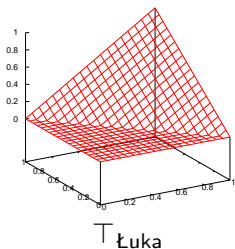
Note that use of product and its dual has nothing to do with probability theory.



The Łukasiewicz t -norm and t -conorm

$$\top_{\text{Łuka}}(x, y) = \max\{0, x + y - 1\}, \quad \perp_{\text{Łuka}}(x, y) = \min\{1, x + y\}$$

$\top_{\text{Łuka}}, \perp_{\text{Łuka}}$ are also called *bold intersection* and *bounded sum*.

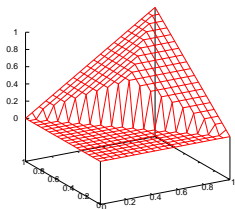
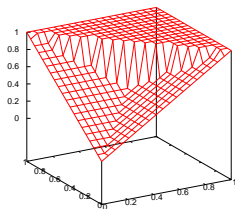


The Nilpotent Minimum and Maximum

$$\top_{\min_0}(x, y) = \begin{cases} \min(x, y) & \text{if } x + y > 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\perp_{\max_1}(x, y) = \begin{cases} \max(x, y) & \text{if } x + y < 1 \\ 1 & \text{otherwise} \end{cases}$$

New since found in 1992 and independently rediscovered in 1995.


 \top_{\min_0}

 \perp_{\max_1}

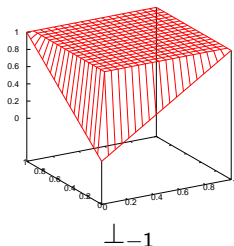
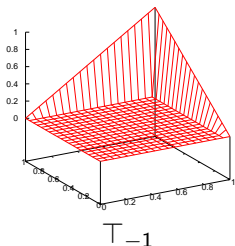
The Drastic Product and Sum

$$\top_{-1}(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1 \\ 0 & \text{otherwise} \end{cases}$$

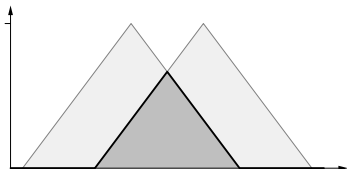
$$\perp_{-1}(x, y) = \begin{cases} \max(x, y) & \text{if } \min(x, y) = 0 \\ 1 & \text{otherwise} \end{cases}$$

\top_{-1} is the weakest t -norm, \perp_{-1} is the strongest t -conorm.

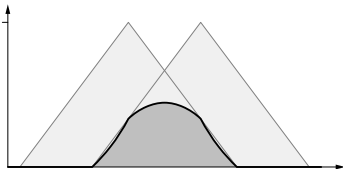
$\top_{-1} \leq \top \leq \top_{\min}$, $\perp_{\max} \leq \perp \leq \perp_{-1}$ for any \top and \perp



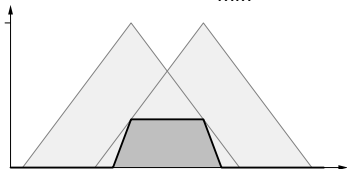
Examples of Fuzzy Intersections



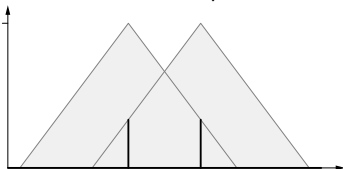
t -norm \top_{\min}



t -norm \top_{prod}



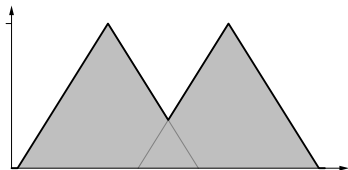
t -norm $\top_{\text{Łuka}}$



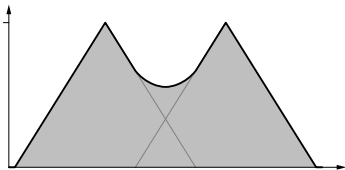
t -norm \top_{-1}

Note that all fuzzy intersections are contained within upper left graph and lower right one.

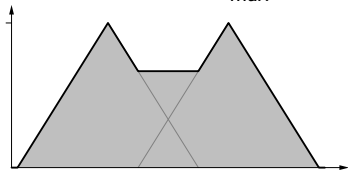
Examples of Fuzzy Unions



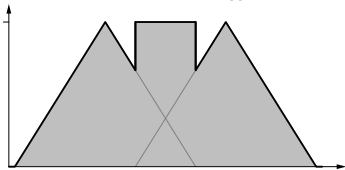
t -conorm \perp_{\max}



t -conorm \perp_{sum}



t -conorm $\perp_{\text{Łuka}}$



t -conorm \perp_{-1}

Note that all fuzzy unions are contained within upper left graph and lower right one.

The Special Role of Minimum and Maximum I

\top_{\min} and \perp_{\max} play key role for intersection and union, resp.

In a practical sense, they are very simple.

Apart from the identity law, commutativity, associativity and monotonicity, they also satisfy the following properties for all $x, y, z \in [0, 1]$:

Distributivity

$$\perp_{\max}(x, \top_{\min}(y, z)) = \top_{\min}(\perp_{\max}(x, y), \perp_{\max}(x, z)),$$

$$\top_{\min}(x, \perp_{\max}(y, z)) = \perp_{\max}(\top_{\min}(x, y), \top_{\min}(x, z))$$

Continuity

\top_{\min} and \perp_{\max} are continuous.

The Special Role of Minimum and Maximum II

Strict monotonicity on the diagonal

$x < y$ implies $\top_{\min}(x, x) < \top_{\min}(y, y)$ and $\perp_{\max}(x, x) < \perp_{\max}(y, y)$.

Idempotency

$$\top_{\min}(x, x) = x, \quad \perp_{\max}(x, x) = x$$

Absorption

$$\top_{\min}(x, \perp_{\max}(x, y)) = x, \quad \perp_{\max}(x, \top_{\min}(x, y)) = x$$

Non-compensation

$x < y < y$ imply $\top_{\min}(x, z) \neq \top_{\min}(y, y)$ and
 $\perp_{\max}(x, z) \neq \perp_{\max}(y, y)$.

The Special Role of Minimum and Maximum III

Is $(\mathcal{F}(X), \top_{\min}, \perp_{\max}, \sim)$ a boolean algebra?

Consider the properties (B1)-(B9) of any Boolean algebra.

For $(\mathcal{F}(X), \top_{\min}, \perp_{\max}, \sim)$ with strong negation \sim only complementary (B7) does not hold.

Hence $(\mathcal{F}(X), \top_{\min}, \perp_{\max}, \sim)$ is a *completely distributive lattice* with identity element μ_X and zero element μ_{\emptyset} .

No lattice $(\mathcal{F}(X), \top, \perp, \sim)$ forms a Boolean algebra

due to the fact that complementary (B7) does not hold:

- There is no complement/negation \sim with $\top(A, \sim A) = \mu_{\emptyset}$.
- There is no complement/negation \sim with $\perp(A, \sim A) = \mu_X$.

Complementary Property of Fuzzy Sets I

Using fuzzy sets, it's **impossible** to keep up a Boolean algebra.

Verify, e.g. that law of contradiction is violated, *i.e.*

$$(\exists x \in X)(A \cap A^c)(x) \neq \emptyset.$$

We use min, max and strong negation \sim as fuzzy set operators.

So we need to show that

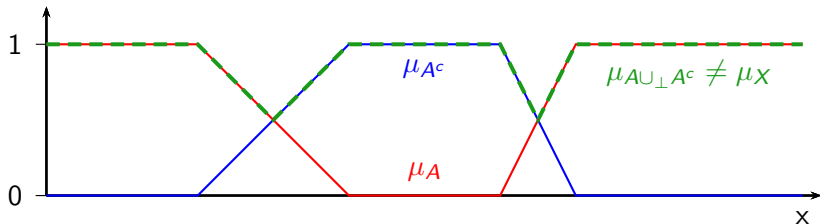
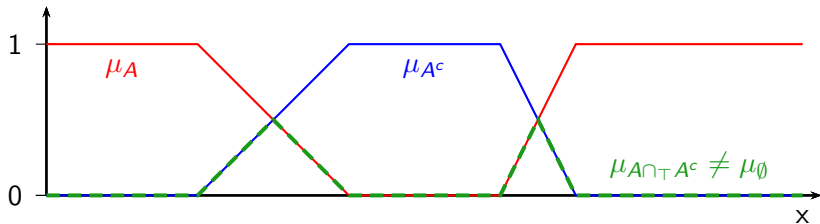
$$\min\{A(x), 1 - A(x)\} = 0$$

is violated for at least one $x \in X$.

easy: This Equation is violated for all $A(x) \in (0, 1)$.

It is satisfied only for $A(x) \in \{0, 1\}$.

Complementary Property of Fuzzy Sets II: Example



The concept of a pseudoinverse

Definition

Let $f : [a, b] \rightarrow [c, d]$ be a monotone function between two closed subintervals of extended real line. The pseudoinverse function to f is the function $f^{(-1)} : [c, d] \rightarrow [a, b]$ defined as

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [a, b] \mid f(x) < y\} & \text{for } f \text{ non-decreasing,} \\ \sup\{x \in [a, b] \mid f(x) > y\} & \text{for } f \text{ non-increasing.} \end{cases}$$

Continuous Archimedean t -norms and t -conorms

broad class of problems relates to representation of multi-place functions by composition of “simpler” functions, e.g.

$$K(x, y) = g(f(x) + f(y))$$

So, one should consider suitable subclass of all t -norms.

Definition

A t -norm \top is

- (a) *continuous* if \top as function is continuous on unit interval,
- (b) *Archimedean* if \top is continuous and $\top(x, x) < x$ for all $x \in]0, 1[$.

Definition

A t -conorm \perp is

- (a) *continuous* if \perp as function is continuous on unit interval,
- (b) *Archimedean* if \perp is continuous and $\perp(x, x) > x$ for all $x \in]0, 1[$.

Continuous Archimedean t -norms

Theorem

A t -norm \top is continuous and Archimedean if and only if there exists a strictly decreasing and continuous function $f : [0, 1] \rightarrow [0, \infty]$ with $f(1) = 0$ such that

$$\top(x, y) = f^{(-1)}(f(x) + f(y)) \quad (1)$$

where

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \text{if } x \leq f(0) \\ 0 & \text{otherwise} \end{cases}$$

is the pseudoinverse of f . Moreover, this representation is unique up to a positive multiplicative constant.

\top is generated by f if \top has representation (1).

f is called *additive generator* of \top .

Additive Generators of t -norms – Examples

Find an additive generator f of $\top_{\text{Łuka}} = \max\{x + y - 1, 0\}$.

for instance $f_{\text{Łuka}}(x) = 1 - x$

then, $f_{\text{Łuka}}^{(-1)}(x) = \max\{1 - x, 0\}$

thus $\top_{\text{Łuka}}(x, y) = f_{\text{Łuka}}^{(-1)}(f_{\text{Łuka}}(x) + f_{\text{Łuka}}(y))$

Find an additive generator f of $\top_{\text{prod}} = x \cdot y$.

to be discussed in the exercise

hint: use of logarithmic and exponential function

Continuous Archimedean t -conorms

Theorem

A t -conorm \perp is continuous and Archimedean if and only if there exists a strictly increasing and continuous function $g : [0, 1] \rightarrow [0, \infty]$ with $g(0) = 0$ such that

$$\perp(x, y) = g^{(-1)}(g(x) + g(y)) \quad (2)$$

where

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } x \leq g(1) \\ 1 & \text{otherwise} \end{cases}$$

is the pseudoinverse of g . Moreover, this representation is unique up to a positive multiplicative constant.

\perp is generated by g if \perp has representation (2).

g is called *additive generator* of \perp .

Additive Generators of t -conorms – Two Examples

Find an additive generator g of $\perp_{\text{Luka}} = \min\{x + y, 1\}$.

for instance $g_{\text{Luka}}(x) = x$

then, $g_{\text{Luka}}^{(-1)}(x) = \min\{x, 1\}$

thus $\perp_{\text{Luka}}(x, y) = g_{\text{Luka}}^{(-1)}(g_{\text{Luka}}(x) + g_{\text{Luka}}(y))$

Find an additive generator g of $\perp_{\text{sum}} = x + y - x \cdot y$.

to be discussed in the exercise

hint: use of logarithmic and exponential function

Now, let us examine some typical families of operations.

Hamacher Family I

$$\top_{\alpha}(x, y) = \frac{x \cdot y}{\alpha + (1 - \alpha)(x + y + x \cdot y)}, \quad \alpha \geq 0,$$

$$\perp_{\beta}(x, y) = \frac{x + y + (\beta - 1) \cdot x \cdot y}{1 + \beta \cdot x \cdot y}, \quad \beta \geq -1,$$

$$\sim_{\gamma}(x) = \frac{1 - x}{1 + \gamma x}, \quad \gamma > -1$$

Theorem

(\top, \perp, \sim) is a De Morgan triplet such that

$$\top(x, y) = \top(x, z) \implies y = z,$$

$$\perp(x, y) = \perp(x, z) \implies y = z,$$

$$\forall z \leq x \exists y, y' \text{ such that } \top(x, y) = z, \perp(z, y') = x$$

and \top and \perp are rational functions if and only if there are numbers $\alpha \geq 0$, $\beta \geq -1$ and $\gamma > -1$ such that $\alpha = \frac{1+\beta}{1+\gamma}$ and $\top = \top_{\alpha}$, $\perp = \perp_{\beta}$ and $\sim = \sim_{\gamma}$.

Hamacher Family II

Additive generators f_α of \top_α are

$$f_\alpha = \begin{cases} \frac{1-x}{x} & \text{if } \alpha = 0 \\ \log \frac{\alpha + (1-\alpha)x}{x} & \text{if } \alpha > 0. \end{cases}$$

Each member of these families is strict t -norm and strict t -conorm, respectively.

Members of this family of t -norms are decreasing functions of parameter α .

Sugeno-Weber Family I

For $\lambda > 1$ and $x, y \in [0, 1]$, define

$$\top_{\lambda}(x, y) = \max \left\{ \frac{x + y - 1 + \lambda xy}{1 + \lambda}, 0 \right\},$$

$$\perp_{\lambda}(x, y) = \min \{x + y + \lambda xy, 1\}.$$

$\lambda = 0$ leads to $\top_{\text{Łuka}}$ and $\perp_{\text{Łuka}}$, resp.

$\lambda \rightarrow \infty$ results in \top_{prod} and \perp_{sum} , resp.

$\lambda \rightarrow -1$ creates \top_{-1} and \perp_{-1} , resp.

Sugeno-Weber Family II

Additive generators f_λ of \top_λ are

$$f_\lambda(x) = \begin{cases} 1 - x & \text{if } \lambda = 0 \\ 1 - \frac{\log(1+\lambda x)}{\log(1+\lambda)} & \text{otherwise.} \end{cases}$$

$\{\top_\lambda\}_{\lambda > -1}$ are increasing functions of parameter λ .

Additive generators of \perp_λ are $g_\lambda(x) = 1 - f_\lambda(x)$.

Yager Family

For $0 < p < \infty$ and $x, y \in [0, 1]$, define

$$\begin{aligned} \top_p(x, y) &= \max \left\{ 1 - ((1-x)^p + (1-y)^p)^{1/p}, 0 \right\}, \\ \perp_p(x, y) &= \min \left\{ (x^p + y^p)^{1/p}, 1 \right\}. \end{aligned}$$

Additive generators of \top_p are

$$f_p(x) = (1-x)^p,$$

and of \perp_p are

$$g_p(x) = x^p.$$

$\{\top_p\}_{0 < p < \infty}$ are strictly increasing in p .

Note that $\lim_{p \rightarrow +\infty} \top_p = \top_{\text{Łuka}}$.

Outline

1. Complement

2. Intersection and Union

3. Implication

S-Implications

R-Implications

QL-Implications

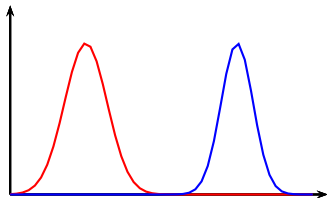
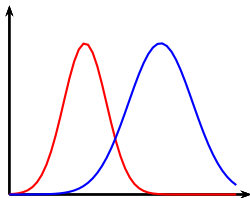
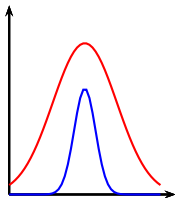
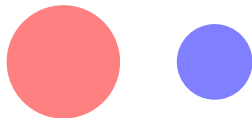
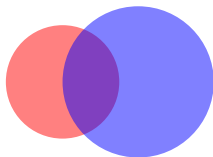
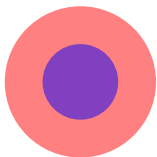
Axioms

List of Fuzzy Implications

Selection of Fuzzy Implications

Fuzzy Implications

crisp: $x \in A \Rightarrow x \in B$, fuzzy: $x \in \mu \Rightarrow x \in \mu'$



Definitions of Fuzzy Implications

One way of defining I is to use $\forall a, b \in \{0, 1\}$

$$I(a, b) = \neg a \vee b.$$

In fuzzy logic, disjunction and negation are t -conorm and fuzzy complement, resp., thus $\forall a, b \in [0, 1]$

$$I(a, b) = \perp(\sim a, b).$$

Another way in classical logic is $\forall a, b \in \{0, 1\}$

$$I(a, b) = \max \{x \in \{0, 1\} \mid a \wedge x \leq b\}.$$

In fuzzy logic, conjunction represents t -norm, thus $\forall a, b \in [0, 1]$

$$I(a, b) = \sup \{x \in [0, 1] \mid \top(a, x) \leq b\}.$$

So, classical definitions are equal, fuzzy extensions are not.

Definitions of Fuzzy Implications

$I(a, b) = \perp(\sim a, b)$ may also be written as either

$$I(a, b) = \neg a \vee (a \wedge b) \quad \text{or}$$

$$I(a, b) = (\neg a \wedge \neg b) \vee b.$$

Fuzzy logical extensions are thus, respectively,

$$I(a, b) = \perp(\sim a, \top(a, b)),$$

$$I(a, b) = \perp(\top(\sim a, \sim b), b)$$

where (\top, \perp, \sim) must be a *De Morgan triplet*.

So again, classical definitions are equal, fuzzy extensions are not.

reason: Law of absorption of negation does not hold in fuzzy logic.

S-Implications

Implications based on $I(a, b) = \perp(\sim a, b)$ are called **S-implications**.

Symbol S is often used to denote t -conorms.

Four well-known S -implications are based on $\sim a = 1 - a$:

Name	$I(a, b)$	$\perp(a, b)$
Kleene-Dienes	$I_{\max}(a, b) = \max(1 - a, b)$	$\max(a, b)$
Reichenbach	$I_{\text{sum}}(a, b) = 1 - a + ab$	$a + b - ab$
Łukasiewicz	$I_{\perp}(a, b) = \min(1, 1 - a + b)$	$\min(1, a + b)$
largest	$I_{-1}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$	$\begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$

S-Implications

The drastic sum \perp_{-1} leads to the largest S-implication I_{-1} due to the following theorem:

Theorem

Let \perp_1, \perp_2 be t-conorms such that $\perp_1(a, b) \leq \perp_2(a, b)$ for all $a, b \in [0, 1]$. Let I_1, I_2 be S-implications based on same fuzzy complement \sim and \perp_1, \perp_2 , respectively. Then $I_1(a, b) \leq I_2(a, b)$ for all $a, b \in [0, 1]$.

Since \perp_{-1} leads to the largest S-implication, similarly, \perp_{\max} leads to the smallest S-implication I_{\max} .

Furthermore,

$$I_{\max} \leq I_{\text{sum}} \leq I_{\perp} \leq I_{-1}.$$

R -Implications

$I(a, b) = \sup \{x \in [0, 1] \mid \top(a, x) \leq b\}$ leads to R -**implications**.

Symbol R represents close connection to residuated semigroup.

Three well-known R -implications are based on $\sim a = 1 - a$:

- Standard fuzzy intersection leads to **Gödel implication**

$$I_{\min}(a, b) = \sup \{x \mid \min(a, x) \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b. \end{cases}$$

- Product leads to **Goguen implication**

$$I_{\text{prod}}(a, b) = \sup \{x \mid ax \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b. \end{cases}$$

- Łukasiewicz t -norm leads to **Łukasiewicz implication**

$$I_{\text{Ł}}(a, b) = \sup \{x \mid \max(0, a + x - 1) \leq b\} = \min(1, 1 - a + b).$$

R-Implications

Name	Formula	$\top(a, b) =$
Gödel	$I_{\min}(a, b) = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b \end{cases}$	$\min(a, b)$
Goguen	$I_{\text{prod}}(a, b) = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b \end{cases}$	ab
Łukasiewicz	$I_{\text{Ł}}(a, b) = \min(1, 1 - a + b)$	$\max(0, a + b - 1)$
largest	$I_{\text{L}}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1, & \text{otherwise} \end{cases}$	not defined

I_{L} is actually the limit of all R -implications.

It serves as least upper bound.

It cannot be defined by $I(a, b) = \sup \{x \in [0, 1] \mid \top(a, x) \leq b\}$.

R-Implications

Theorem

Let \top_1, \top_2 be t -norms such that $\top_1(a, b) \leq \top_2(a, b)$ for all $a, b \in [0, 1]$. Let I_1, I_2 be R -implications based on \top_1, \top_2 , respectively. Then $I_1(a, b) \geq I_2(a, b)$ for all $a, b \in [0, 1]$.

It follows that Gödel I_{\min} is the smallest R -implication.

Furthermore,

$$I_{\min} \leq I_{\text{prod}} \leq I_{\text{L}} \leq I_{\text{L}}.$$

QL-Implications

Implications based on $I(a, b) = \perp(\sim a, \top(a, b))$ are called **QL-implications** (QL from quantum logic).

Four well-known QL-implications are based on $\sim a = 1 - a$:

- Standard min and max lead to **Zadeh implication**

$$I_Z(a, b) = \max[1 - a, \min(a, b)].$$

- The algebraic product and sum lead to

$$I_p(a, b) = 1 - a + a^2 b.$$

- Using \top_{\perp} and \perp_{\perp} leads to **Kleene-Dienes implication** again.
- Using \top_{-1} and \perp_{-1} leads to

$$I_q(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } a \neq 1, b \neq 1 \\ 1, & \text{if } a \neq 1, b = 1. \end{cases}$$

Axioms

All I come from generalizations of the classical implication.

They collapse to the classical implication when truth values are 0 or 1.

Generalizing classical properties leads to following axioms:

- $a \leq b$ implies $I(a, x) \geq I(b, x)$ (*monotonicity in 1st argument*)
- $a \leq b$ implies $I(x, a) \leq I(x, b)$ (*monotonicity in 2nd argument*)
- $I(0, a) = 1$ (*dominance of falsity*)
- $I(1, b) = b$ (*neutrality of truth*)
- $I(a, a) = 1$ (*identity*)
- $I(a, I(b, c)) = I(b, I(a, c))$ (*exchange property*)
- $I(a, b) = 1$ if and only if $a \leq b$ (*boundary condition*)
- $I(a, b) = I(\sim b, \sim a)$ for fuzzy complement \sim (*contraposition*)
- I is a continuous function (*continuity*)

Generator Function

I that satisfy all listed axioms are characterized by this theorem:

Theorem

A function $I : [0, 1]^2 \rightarrow [0, 1]$ satisfies Axioms 1–9 of fuzzy implications for a particular fuzzy complement \sim if and only if there exists a strict increasing continuous function $f : [0, 1] \rightarrow [0, \infty)$ such that $f(0) = 0$,

$$I(a, b) = f^{(-1)}(f(1) - f(a) + f(b))$$

for all $a, b \in [0, 1]$, and

$$\sim a = f^{-1}(f(1) - f(a))$$

for all $a \in [0, 1]$.

Example

Consider $f_\lambda(a) = \ln(1 + \lambda a)$ with $a \in [0, 1]$ and $\lambda > 0$.

Its pseudo-inverse is

$$f_\lambda^{(-1)}(a) = \begin{cases} \frac{e^a - 1}{\lambda}, & \text{if } 0 \leq a \leq \ln(1 + \lambda) \\ 1, & \text{otherwise.} \end{cases}$$

The fuzzy complement generated by f for all $a \in [0, 1]$ is

$$n_\lambda(a) = \frac{1 - a}{1 + \lambda a}.$$

The resulting fuzzy implication for all $a, b \in [0, 1]$ is thus

$$I_\lambda(a, b) = \min \left(1, \frac{1 - a + b + \lambda b}{1 + \lambda a} \right).$$

If $\lambda \in (-1, 0)$, then I_λ is called **pseudo-Lukasiewicz implication**.

List of Fuzzy Implications

Name	Class	Form $I(a, b) =$	Axioms	Complement
Gaines-Rescher		$\begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$	1-8	$1 - a$
Gödel	R	$\begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$	1-7	
Goguen	R	$\begin{cases} 1 & \text{if } a \leq b \\ b/a & \text{otherwise} \end{cases}$	1-7, 9	
Kleene-Dienes	S, QL	$\max(1 - a, b)$	1-4, 6, 8, 9	$1 - a$
Łukasiewicz	R, S	$\min(1, 1 - a + b)$	1-9	$1 - a$
Pseudo-Łukasiewicz 1	R, S	$\min \left[1, \frac{1-a+(1+\lambda)b}{1+\lambda a} \right]$	1-9	$\frac{1-a}{1+\lambda a}, (\lambda > -1)$
Pseudo-Łukasiewicz 2	R, S	$\min [1, 1 - a^w + b^w]$	1-9	$(1 - a^w)^{\frac{1}{w}}, (w > 0)$
Reichenbach	S	$1 - a + ab$	1-4, 6, 8, 9	$1 - a$
Wu		$\begin{cases} 1 & \text{if } a \leq b \\ \min(1 - a, b) & \text{otherwise} \end{cases}$	1-3,5,7,8	$1 - a$
Zadeh	QL	$\max[1 - a, \min(a, b)]$	1-4, 9	$1 - a$

Which Fuzzy Implication?

Since the meaning of I is not unique, we must resolve the following question:

Which I should be used for calculating the fuzzy relation R ?

Hence meaningful criteria are needed.

They emerge from various fuzzy inference rules, *i.e.* modus ponens, modus tollens, hypothetical syllogism.