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INF

FAKULTÄT FÜR
INFORMATIK

Fuzzy Systems

Fuzzy Arithmetic

Prof. Dr. Rudolf Kruse **Christoph Doell**

{kruse,cmoewes}@iws.cs.uni-magdeburg.de
Otto-von-Guericke University of Magdeburg
Faculty of Computer Science
Department of Knowledge Processing and Language Engineering

Outline

1. **The Extension Principle**

   Truth Values

   Extensions to Sets and Fuzzy Sets

2. **Fuzzy Arithmetic**

Motivation I

How to extend $\phi : X^n \to Y$ to $\hat{\phi} : \mathcal{F}(X)^n \to \mathcal{F}(Y)$?

Let $\mu \in \mathcal{F}(\mathbb{R})$ be a fuzzy set of the imprecise concept "about 2".

Then the degree of membership $\mu(2.2)$ can be seen as *truth value* of the statement "2.2 is about equal to 2".

Let $\mu' \in \mathcal{F}(\mathbb{R})$ be a fuzzy set of the imprecise concept "old".

Then the truth value of "2.2 is about equal 2 **and** 2.2 is old" can be seen as membership degree of 2.2 *w.r.t.* imprecise concept "about 2 and old".

Motivation II – Operating on Truth Values

Any $\top$ ($\perp$) can be used to represent conjunction (disjunction).

However, now only $T_{\min}$ and $\perp_{\max}$ shall be used.

Let $\mathcal{P}$ be set of imprecise statements that can be combined by *and*, *or*.

truth : $\mathcal{P} \to [0, 1]$ assigns truth value truth(a) to every $a \in \mathcal{P}$.

truth(a) $= 0$ means a is definitely false.

truth(a) $= 1$ means a is definitely true.

If $0 <$ truth(a) < 1, then only gradual truth of statement a.

Motivation III – Extension Principle

Combination of two statements $a, b \in P$:

$$\begin{aligned} \text{truth}(a \text{ and } b) = \quad & \text{truth}(a \wedge b) = \quad && \min\{\text{truth}(a), \text{truth}(b)\}, \\ \text{truth}(a \text{ or } b) = \quad & \text{truth}(a \vee b) = \quad && \max\{\text{truth}(a), \text{truth}(b)\} \end{aligned}$$

For infinite number of statements $a_i, i \in I$:

$$\begin{aligned} \text{truth}(\forall i \in I : a_i) &= \inf\left\{\text{truth}(a_i) \mid i \in I\right\}, \\ \text{truth}(\exists i \in I : a_i) &= \sup\left\{\text{truth}(a_i) \mid i \in I\right\} \end{aligned}$$

This concept helps to extend $\phi : X^n \to Y$ to $\hat{\phi} : \mathcal{F}(X)^n \to \mathcal{F}(Y)$.

- Crisp tuple $(x_1, \ldots, x_n)$ is mapped to crisp value $\phi(x_1, \ldots, x_n)$.
- Imprecise descriptions $(\mu_1, \ldots, \mu_n)$ of $(x_1, \ldots, x_n)$ are mapped to fuzzy value $\hat{\phi}(\mu_1, \ldots, \mu_n)$.

Example – How to extend the addition?

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (a, b) \mapsto a + b$

Extensions to sets: $+ : 2^{\mathbb{R}} \times 2^{\mathbb{R}} \to 2^{\mathbb{R}}$

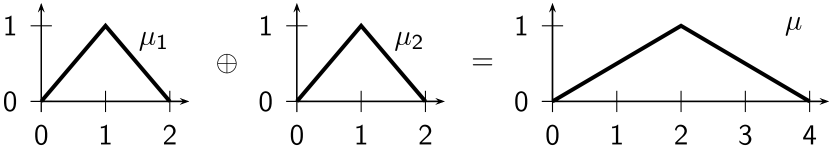
$$(A, B) \mapsto A + B = \{y \mid (\exists a)(\exists b) y = a + b \wedge a \in A \wedge b \in B\}$$

Extensions to fuzzy sets:

$$+ : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R}), \quad (\mu_1, \mu_2) \mapsto \mu_1 \oplus \mu_2$$

$$\begin{aligned}
\text{truth}(y \in \mu_1 \oplus \mu_2) &= \text{truth}((\exists a)(\exists b) : y = a + b \wedge a \in \mu_1 \wedge b \in \mu_2) \\
&= \sup_{a,b}\{\text{truth}(y = a + b) \wedge \text{truth}(a \in \mu_1) \wedge \\
&\qquad \text{truth}(b \in \mu_2)\} \\
&= \sup_{a,b:y=a+b}\{\min(\mu_1(a),\ \mu_2(b))\}
\end{aligned}$$

Example – How to extend the addition?



$\mu(2) = 1$ because $\mu_1(1) = 1$ and $\mu_2(1) = 1$

$\mu(5) = 0$ because if $a + b = 5$, then $\min\{\mu_1(a),\ \mu_2(b)\} = 0$

$\mu(1) = 0.5$ because it is the result of an optimization task with optimum at, *e.g.* $a = 0.5$ and $b = 0.5$

Extension to Sets

Definition

Let $\phi : X^n \to Y$ be a mapping. The *extension* $\hat{\phi}$ of ϕ is given by

$$\hat{\phi} : [2^X]^n \to 2^Y \quad \text{with}$$

$$\hat{\phi}(A_1, \ldots, A_n) = \{y \in Y \mid \exists (x_1, \ldots, x_n) \in A_1 \times \ldots \times A_n : \phi(x_1, \ldots, x_n) = y\}.$$

Extension to Fuzzy Sets

Definition

Let $\phi : X^n \to Y$ be a mapping. The *extension* $\hat{\phi}$ of ϕ is given by

$$\hat{\phi} : [\mathcal{F}(X)]^n \to \mathcal{F}(Y) \quad \text{with}$$

$$\hat{\phi}(\mu_1, \dots, \mu_n)(y) = \sup\{\min\{\mu_1(x_1), \dots, \mu_n(x_n)\} \mid (x_1, \dots, x_n) \in X^n \wedge \phi(x_1, \dots, x_n) = y\}$$

assuming that $\sup \emptyset = 0$.

Example I

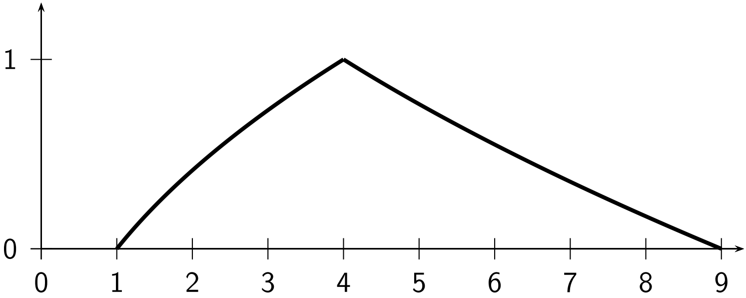
Let fuzzy set "approximately 2" be defined as

$$\mu(x) = \begin{cases} x-1, & \text{if } 1 \le x \le 2 \\ 3-x, & \text{if } 2 \le x \le 3 \\ 0, & \text{otherwise.} \end{cases}$$

The extension of $\phi : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ to fuzzy sets on $\mathbb{R}$ is

$$\begin{aligned} \hat{\phi}(\mu)(y) &= \sup\left\{ \mu(x) \;\middle|\; x \in \mathbb{R} \wedge x^2 = y \right\} \\ &= \begin{cases} \sqrt{y}-1, & \text{if } 1 \le y \le 4 \\ 3-\sqrt{y}, & \text{if } 4 \le y \le 9 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Example II



The extension principle is taken as basis for “fuzzifying” whole theories.

Now, it will be applied to arithmetic operations on fuzzy intervals.

Outline

Fuzzy Sets on the Real Numbers

There are many different types of fuzzy sets.

Very interesting are fuzzy sets defined on set $\mathbb{R}$ of real numbers.

Membership functions of such sets, *i.e.*

$$\mu : \mathbb{R} \to [0, 1],$$

clearly indicate quantitative meaning.

Such concepts may essentially characterize states of fuzzy variables.

They play important role in many applications, *e.g.* fuzzy control, decision making, approximate reasoning, optimization, and statistics with imprecise probabilities.

Some Special Fuzzy Sets I

Here, we only consider special classes $\mathcal{F}(\mathbb{R})$ of fuzzy sets μ on $\mathbb{R}$.

Definition

(a) $\mathcal{F}_N(\mathbb{R}) \overset{\text{def}}{=} \{\mu \in \mathcal{F}(\mathbb{R}) \mid \exists x \in \mathbb{R} : \mu(x) = 1\}\,,$

(b) $\mathcal{F}_C(\mathbb{R}) \overset{\text{def}}{=} \{\mu \in \mathcal{F}_N(\mathbb{R}) \mid \forall \alpha \in (0,1] : [\mu]_\alpha \text{ is compact }\}\,,$

(c) $\mathcal{F}_I(\mathbb{R}) \overset{\text{def}}{=} \{\mu \in \mathcal{F}_N(\mathbb{R}) \mid \forall a, b, c \in \mathbb{R} : c \in [a, b] \Rightarrow$

$\mu(c) \geq \min\{\mu(a), \mu(b)\}\,\}.$

Some Special Fuzzy Sets II

An element in $\mathcal{F}_N(\mathbb{R})$ is called **normal fuzzy set**:

- It's meaningful if $\mu \in \mathcal{F}_N(\mathbb{R})$ is used as *imprecise description* of an existing (but not precisely measurable) variable $\subseteq \mathbb{R}$.
- In such cases it would not be plausible to assign maximum membership degree of 1 to no single real number.

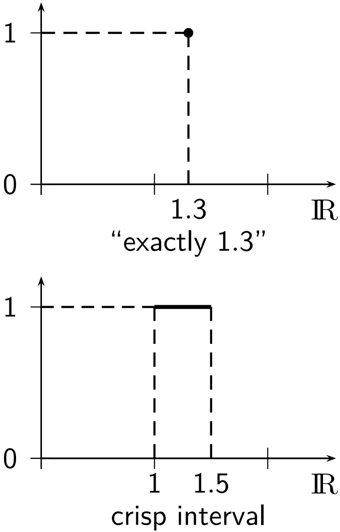
Sets in $\mathcal{F}_C(\mathbb{R})$ are **upper semi-continuous**:

- Function f is upper semi-continuous at point x_0 if values near x_0 are either close to $f(x_0)$ or less than $f(x_0)$
  $\Rightarrow \lim_{x \to x_0} \sup f(x) \leq f(x_0)$.
- This simplifies arithmetic operations applied to them.

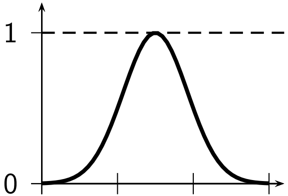
Fuzzy sets in $\mathcal{F}_I(\mathbb{R})$ are called **fuzzy intervals**:

- The are *normal* and *fuzzy convex*.
- Their core is a classical interval.
- $\mu \in \mathcal{F}_I(\mathbb{R})$ for real numbers are called **fuzzy numbers**.

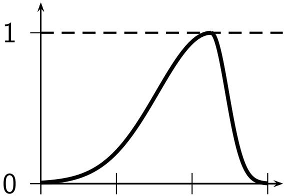
Comparison of Crisp Sets and Fuzzy Sets on $\mathbb{R}$



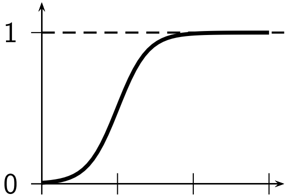
Basic Types of Fuzzy Numbers



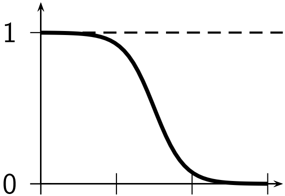
"around r"
symmetric bell-shaped



"around r"
asymmetric bell-shaped



"large number"
right-open sigmoid



"small number"
left-open sigmoid

Quantitative Fuzzy Variables

The concept of a fuzzy number plays fundamental role in formulating *quantitative fuzzy variables*.

These are variables whose states are fuzzy numbers.

When the fuzzy numbers represent linguistic concepts, *e.g.*

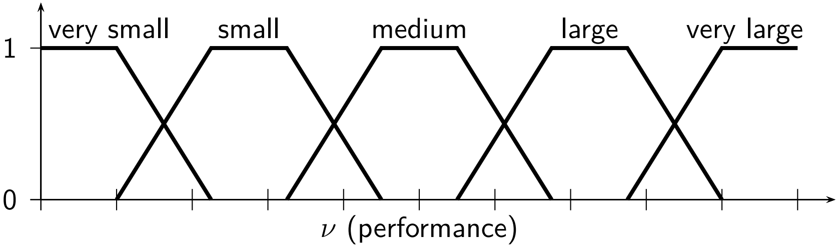
very small, *small*, *medium*, etc.

then final constructs are called **linguistic variables**.

Each linguistic variable is defined in terms of *base variable* which is a variable in classical sense, *e.g.* temperature, pressure, age.

Linguistic terms representing approximate values of base variable are captured by appropriate fuzzy numbers.

Linguistic Variables



Each linguistic variable is defined by quintuple (ν, T, X, g, m).

- *name* ν of the variable
- set T of *linguistic terms* of ν
- *base variable* $X \subseteq \mathbb{R}$
- *syntactic rule* g (grammar) for generating linguistic terms
- *semantic rule* m that assigns *meaning* $m(t)$ to every $t \in T$, *i.e.* $m : T \to \mathcal{F}(X)$

Operations on Linguistic Variables

To deal with linguistic variables, consider

- not only set-theoretic operations
- but also arithmetic operations on fuzzy numbers (*i.e.* interval arithmetic).

e.g. statistics:

- Given a sample $=$ (*small*, *medium*, *small*, *large*, ...).
- How to define mean value or standard deviation?

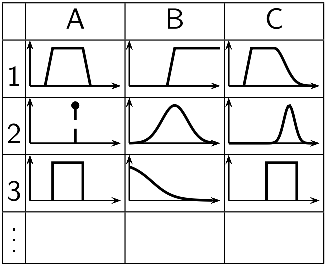
Analysis of Linguistic Data

Linguistic Data

	A	B	C
1	large	very large	medium
2	2.5	medium	about 7
3	[3, 4]	small	[7, 8]
⋮			

linguistic modeling

Fuzzy Data



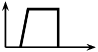
computing with words

statistics with fuzzy sets

linguistic approximation

"The mean *w.r.t.* A is approximatly 4."

mean of attribute A



Example – Application of Linguistic Data

Consider the problem to model the climatic conditions of several towns.

A tourist may want information about tourist attractions.

Assume that linguistic random samples are based on subjective observations of selected people, *e.g.*

- climatic attribute *clouding*
- linguistic values *cloudless, clear, fair, cloudy, . . .*

Example – Linguistic Modeling by an Expert

The attribute *clouding* is modeled by elementary linguistic values, *e.g.*

$$\begin{aligned}
\text{cloudless} &\mapsto \text{sigmoid}(0, -0.07)\\
\text{clear} &\mapsto \text{Gauss}(25, 15)\\
\text{fair} &\mapsto \text{Gauss}(50, 20)\\
\text{cloudy} &\mapsto \text{Gauss}(75, 15)\\
\text{overcast} &\mapsto \text{sigmoid}(100, 0.07)\\
\text{exactly)}(x) &\mapsto \text{exact}(x)\\
\text{approx)}(x) &\mapsto \text{Gauss}(x, 3)\\
\text{between}(x, y) &\mapsto \text{rectangle}(x, y)\\
\text{approx_between}(x, y) &\mapsto \text{trapezoid}(x - 20, x, y, y + 20)
\end{aligned}$$

where $x, y \in [0, 100] \subset \mathbb{R}$.

Example

Gauss(a, b) is, *e.g.* a function defined by

$$f(x) = \exp\left(-\left(\frac{x-a}{b}\right)^2\right), \qquad x, a, b \in \mathbb{R}, \quad b > 0$$

induced language of expressions:

$$\begin{aligned} \langle\text{expression}\rangle \;:=\; & \langle\text{elementary linguistic value}\rangle \mid \\ & (\ \langle\text{expression}\rangle\ ) \mid \\ & \{\ \text{not} \mid \text{dil} \mid \text{con} \mid \text{int}\ \}\ \langle\text{expression}\rangle \mid \\ & \langle\text{expression}\rangle\ \{\ \text{and} \mid \text{or}\ \}\ \langle\text{expression}\rangle\ , \end{aligned}$$

e.g. approx(x) *and cloudy* is represented by function

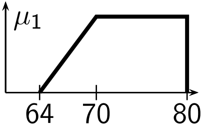
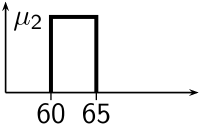
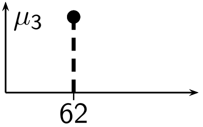
$$\min\{\text{Gauss}(x, 3), \text{Gauss}(75, 15)\}\,.$$

Example – Linguistic Random Sample

Attribute	:	Clouding
Observations	:	Limassol, Cyprus
2009/10/23	:	cloudy
2009/10/24	:	dil approx_between(50, 70)
2009/10/25	:	fair or cloudy
2009/10/26	:	approx(75)
2009/10/27	:	dil(clear or fair)
2009/10/28	:	int cloudy
2009/10/29	:	con fair
2009/11/30	:	approx(0)
2009/11/31	:	cloudless
2009/11/01	:	cloudless or dil clear
2009/11/02	:	overcast
2009/11/03	:	cloudy and between(70, 80)
...	:	...
2009/11/10	:	clear

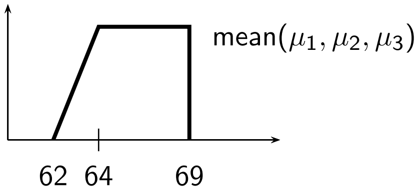
Statistics with fuzzy sets are necessary to analyze linguistic data.

Example – Ling. Random Sample of 3 People

no.	age (linguistic data)	age (fuzzy data)
1	approx. between 70 and 80 and definitely not older than 80	
2	between 60 and 65	
3	62	

Example – Mean Value of Ling. Random Sample

$$\text{mean}(\mu_1, \mu_2, \mu_3) = \frac{1}{3}\,(\mu_1 \oplus \mu_2 \oplus \mu_3)$$



i.e. approximately between 64 and 69 but not older than 69

Efficient Operations I

How to define arithmetic operations for calculating with $\mathcal{F}(\mathbb{R})$?

Using extension principle for sum $\mu \oplus \mu'$, product $\mu \odot \mu'$ and reciprocal value $\mathrm{rec}(\mu)$ of arbitrary fuzzy sets $\mu, \mu' \in \mathcal{F}(\mathbb{R})$

$$
\begin{aligned}
(\mu \oplus \mu')(t) &= \sup\left\{\min\{\mu(x_1), \mu'(x_2)\} \mid x_1, x_2 \in \mathbb{R}, x_1 + x_2 = t\right\},\\
(\mu \odot \mu')(t) &= \sup\left\{\min\{\mu(x_1), \mu'(x_2)\} \mid x_1, x_2 \in \mathbb{R}, x_1 \cdot x_2 = t\right\},\\
\mathrm{rec}(\mu)(t) &= \sup\left\{\mu(x) \;\middle|\; x \in \mathbb{R} \setminus \{0\}, \frac{1}{x} = t\right\}.
\end{aligned}
$$

In general, operations on fuzzy sets are much more complicated (especially if vertical instead of horizontal representation is applied).

It's desirable to reduce fuzzy arithmetic to ordinary set arithmetic.

Then, we apply elementary operations of *interval arithmetic*.

Efficient Operations II

Definition

A family $(A_\alpha)_{\alpha\in(0,1)}$ of sets is called *set representation* of $\mu \in \mathcal{F}_N(\mathbb{R})$ if

(a) $0 < \alpha < \beta < 1 \Longrightarrow A_\beta \subseteq A_\alpha \subseteq \mathbb{R}$ and

(b) $\mu(t) = \sup\{\alpha \in [0,1] \mid t \in A_\alpha\}$

holds where $\sup \emptyset = 0$.

Theorem

Let $\mu \in F_N(\mathbb{R})$. The family $(A_\alpha)_{\alpha\in(0,1)}$ of sets is a set representation of μ if and only if

$$[\mu]_{\underline{\alpha}} = \{t \in \mathbb{R} \mid \mu(t) > \alpha\} \subseteq A_\alpha \subseteq \{t \in \mathbb{R} \mid \mu(t) \geq \alpha\} = [\mu]_\alpha$$

is valid for all $\alpha \in (0,1)$.

Efficient Operations III

Theorem

Let $\mu_1, \mu_2, \ldots, \mu_n$ be normal fuzzy sets of $\mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ be a mapping. Then the following holds:

(a) $\forall \alpha \in [0, 1) : [\hat{\phi}(\mu_1, \ldots, \mu_n)]_{\underline{\alpha}} = \phi([\mu_1]_{\underline{\alpha}}, \ldots, [\mu_n]_{\underline{\alpha}})$,

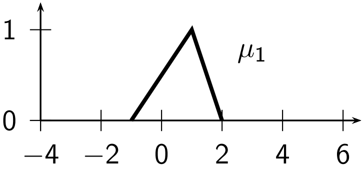
(b) $\forall \alpha \in (0, 1] : [\hat{\phi}(\mu_1, \ldots, \mu_n)]_{\alpha} \supseteq \phi([\mu_1]_{\alpha}, \ldots, [\mu_n]_{\alpha})$,

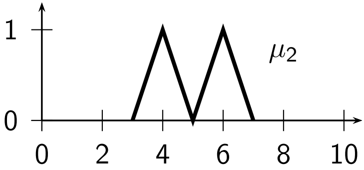
(c) if $((A_i)_\alpha)_{\alpha \in (0,1)}$ *is a set representation of* μ_i *for* $1 \leq i \leq n$*, then* $(\phi((A_1)_\alpha, \ldots, (A_n)_\alpha))_{\alpha \in (0,1)}$ *is a set representation of* $\hat{\phi}(\mu_1, \ldots, \mu_n)$.

For arbitrary mapping ϕ, set representation of its extension $\hat{\phi}$ can be obtained with help of set representation $((A_i)_\alpha)_{\alpha \in (0,1)}, i = 1, 2, \ldots, n$.

It's used to carry out arithmetic operations on fuzzy sets efficiently.

Example I





For μ_1, μ_2, the set representations are

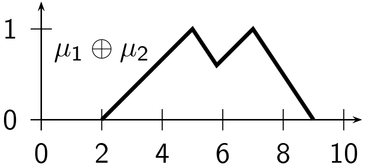
- $[\mu_1]_\alpha = [2\alpha - 1, 2 - \alpha]$,
- $[\mu_2]_\alpha = [\alpha + 3, 5 - \alpha] \cup [\alpha + 5, 7 - \alpha]$.

Let $\text{add}(x, y) = x + y$, then $(A_\alpha)_{\alpha \in (0,1)}$ represents $\mu_1 \oplus \mu_2$

$$A_\alpha = \text{add}([\mu_1]_\alpha, [\mu_2]_\alpha) = [3\alpha + 2, 7 - 2\alpha] \cup [3\alpha + 4, 9 - 2\alpha]$$

$$= \begin{cases} [3\alpha + 2, 7 - 2\alpha] \cup [3\alpha + 4, 9 - 2\alpha], & \text{if } \alpha \geq 0.6 \\ [3\alpha + 2, 9 - 2\alpha], & \text{if } \alpha \leq 0.6. \end{cases}$$

Example II



$$(\mu_1 \oplus \mu_2)(x) = \begin{cases} \frac{x-2}{3}, & \text{if } 2 \leq x \leq 5 \\ \frac{7-x}{2}, & \text{if } 5 \leq x \leq 5.8 \\ \frac{x-4}{3}, & \text{if } 5.8 \leq x \leq 7 \\ \frac{9-x}{2}, & \text{if } 7 \leq x \leq 9 \\ 0, & \text{otherwise} \end{cases}$$

Interval Arithmetic I

Determining the set representations of arbitrary combinations of fuzzy sets can be reduced very often to simple interval arithmetic.

Using fundamental operations of arithmetic leads to the following ($a, b, c, d \in \mathbb{R}$):

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b] \cdot [c, d] = \begin{cases} [ac, bd], & \text{for } a \geq 0 \wedge c \geq 0 \\ [bd, ac], & \text{for } b < 0 \wedge d < 0 \\ [\min\{ad, bc\}, \max\{ad, bc\}], & \text{for } ab \geq 0 \wedge cd \geq 0 \wedge ac < 0 \\ [\min\{ad, bc\}, \max\{ac, bd\}], & \text{for } ab < 0 \vee cd < 0 \end{cases}$$

$$\frac{1}{[a, b]} = \begin{cases} \left[\frac{1}{b}, \frac{1}{a}\right], & \text{if } 0 \notin [a, b] \\ \left[\frac{1}{b}, \infty\right) \cup \left(-\infty, \frac{1}{a}\right], & \text{if } a < 0 \wedge b > 0 \\ \left[\frac{1}{b}, \infty\right), & \text{if } a = 0 \wedge b > 0 \\ \left(-\infty, \frac{1}{a}\right], & \text{if } a < 0 \wedge b = 0 \end{cases}$$

Interval Arithmetic II

In general, set representation of α-cuts of extensions $\hat{\phi}(\mu_1, \dots, \mu_n)$ cannot be determined directly from α-cuts.

It only works always for continuous ϕ and fuzzy sets in $\mathcal{F}_C(\mathbb{R})$.

Theorem

Let $\mu_1, \mu_2, \dots, \mu_n \in \mathcal{F}_C(\mathbb{R})$ *and* $\phi : \mathbb{R}^n \to \mathbb{R}$ *be a continuous mapping. Then*

$$\forall \alpha \in (0, 1] : [\hat{\phi}(\mu_1, \dots, \mu_n)]_\alpha = \phi([\mu_1]_\alpha, \dots, [\mu_n]_\alpha).$$

So, a horizontal representation is better than a vertical one.
Finding $\hat{\phi}$ values is easier than directly applying the extension principle.
However, all α-cuts cannot be stored in a computer.
Only a finite number of α-cuts can be stored.