## 9. Similarity Relations

**Example 9.1**

Specification of a partial control mapping ("good control actions")

<table>
<thead>
<tr>
<th>gradient</th>
<th>-40</th>
<th>-6</th>
<th>-3</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>-70</td>
<td>22,5</td>
<td>15,0</td>
<td>15,0</td>
<td>10,0</td>
<td>10,0</td>
<td>5,0</td>
<td>5,0</td>
</tr>
<tr>
<td>-50</td>
<td>22,5</td>
<td>15,0</td>
<td>10,0</td>
<td>10,0</td>
<td>5,0</td>
<td>5,0</td>
<td>0,0</td>
</tr>
<tr>
<td>-30</td>
<td>15,0</td>
<td>10,0</td>
<td>50,0</td>
<td>5,0</td>
<td>0,0</td>
<td>0,0</td>
<td>0,0</td>
</tr>
<tr>
<td>0</td>
<td>5,0</td>
<td>5,0</td>
<td>0,0</td>
<td>0,0</td>
<td>0,0</td>
<td>-10,0</td>
<td>-15,0</td>
</tr>
<tr>
<td>30</td>
<td>0,0</td>
<td>0,0</td>
<td>0,0</td>
<td>0,0</td>
<td>-5,0</td>
<td>-5,0</td>
<td>-10,0</td>
</tr>
<tr>
<td>50</td>
<td>0,0</td>
<td>-5,0</td>
<td>-5,0</td>
<td>-10,0</td>
<td>-15,0</td>
<td>-15,0</td>
<td>-22,5</td>
</tr>
<tr>
<td>70</td>
<td>-5,0</td>
<td>-5,0</td>
<td>-15,0</td>
<td>-15,0</td>
<td>-15,0</td>
<td>-15,0</td>
<td>-15,0</td>
</tr>
</tbody>
</table>
Interpolation of this table

Additional knowledge was available: Some values are indistinguishable (from a measurement point of view) or should be treated in a similar way.

Problem 1: How to model such similarity information?

Problem 2: How to interpolate in the case of existing similarity information?
How to model similarity?

Proposal 1: equivalence relation

(I) \( x \approx x \) (reflexivity)
(II) \( x \approx y \iff y \approx x \) (symmetry)
(III) \( x \approx y \land y \approx z \rightarrow x \approx z \) (transitivity)

- \( x \) and \( y \) similar (\( x \approx y \)), if and only if \( |x-y|<\varepsilon \), (\( \varepsilon \) fixed)
  \( \approx \) is not transitive, Poincaré paradox
  \( x\approx y, y\approx z, x\not\approx z \)
- counterintuitive!
Proposal 2: similarity relation (multi-valued equivalence relation)

[x\approx y] degree, to which x\approx y holds

\[ E_\sim: X \times X \to [0,1], (x,y) \to [x\approx y] \]

1. \[ E_\sim(x,x) = 1 \]
2. \[ E_\sim(x,y) = E_\sim(y,x) \]
3. \[ \Pi(E_\sim(x,y), E_\sim(y,z)) \leq E_\sim(x,z), \] where \( \Pi \) is a \( t \)-norm

\( E_\sim \) is called fuzzy equality relation, similarity relation, indistinguishability operator or tolerance relation.
**Example**

δ pseudo metric on \( X \)

\[ \Pi(\alpha,\beta) = \max\{\alpha + \beta - 1, 0\} \] Lukasiewicz \( t \)-norm,

**Then**

\[ E_\delta (x,y) = 1 - \min\{\delta(x,y), 1\} \] fuzzy equality relation

\( \delta_E(x,y) = 1 - E(x,y) \) induced pseudo metric

i.e. fuzzy equality and distance are dual notions

**Formal Definition**

\( E: X \times X \rightarrow [0,1] \) similarity relation, iff

1. \( E(x,x) = 1 \)
2. \( E(x,y) = E(y,x) \)
3. \( \max\{(E(x,y) + E(y,z) - 1), 0\} \leq E(x,z) \)
Fuzzy Sets as a Derived Concept

\[ \delta(x, y) = |x - y| \quad \text{metric} \]
\[ E_\delta(x, y) = 1 - \min \{|x - y|, 1\} \quad \text{similarity relation} \]

\[ \mu_y : X \rightarrow [0, 1] \]

\[ x \rightarrow E_\delta(x, y) \text{ fuzzy singleton} \]

\[ \mu_y \text{ describes the “local“ similarities} \]
**Example 9.2**

\[ E : \mathbb{R} \times \mathbb{R} \rightarrow [0,1], (x, y) \mapsto 1 - \min \{ |x - y|, 1 \} \]

is a similarity relation w.r.t. the t-norm \( T_{Luka} \)

**Def. 9.3**

Let \( E \) be a similarity relation on \( X \) w.r.t. \( T \).

\[ \mu \in F(X) \text{ is extensional iff } T (\mu(x), E(x, y)) \leq \mu(y) \text{ for all } x, y. \]

**Def. 9.4**

Let \( E \) be a similarity relation on \( X \) w.r.t. \( T \). And \( M \subseteq X \).

\[ \mu_M : X \rightarrow [0,1], x \mapsto \sup \{ E(x, x') \mid x' \in M \} \]

Is called extensional hull of \( M \).

**Example 9.5**

A singleton is the extensional hull of \( \{x_0\} \).
Def. 9.6
Let $E$, $F$ be similarity relations on $X$ and $Y$. 
$\zeta: \! X \rightarrow Y$ is extensional with respect to $E$, $F$ iff $E(x, x') \leq F(\zeta(x), \zeta(x'))$ holds.

Theorem 9.7
Let $E_1, \ldots, E_n$ similarity relations w.r.t. $\prod$ on $X_1, \ldots, X_n$.
Define $E: (X_1 \times \cdots \times X_n)^2 \rightarrow [0,1]$,
$\{(x_1, \ldots, x_n), (y_1, \ldots, y_n)\} \mapsto \min\{E_1(x_1, y_1), \ldots, E_n(x_n, y_n)\}$
a) $E$ is a similarity relation w.r.t. $\prod$ on $X_1 \times \cdots \times X_n$.
b) For all $i \in \{1, \ldots, n\}$ the projection $\pi_i: X_1 \times \cdots \times X_n \rightarrow X_i$ is extensional w.r.t. $E$ and $E_i$.
c) If $E'$ is a similarity relation w.r.t. $\prod$ on $X_1 \times \cdots \times X_n$, i.e. all projections are extensional, then $E' \leq E$ holds.

Remark 9.8
$E$ is the biggest similarity relation for which all projections are extensional.
Specification of similarity relations

**given** a family of fuzzy sets that describes the “local“ similarities

then

- there is a similarity relation on D with induced fuzzy sets $\mu_i$, iff

$$\sup_{x \in X} \{\mu_i(x) + \mu_j(x) - 1\} \leq \inf_{y \in X} \{1 - |\mu_i(y) - \mu_j(y)|\} \text{ for all } i, j$$

- if $\mu_i(x) + \mu_j(x) \leq 1$ for $i \neq j$ then there is a similarity relation $E$ on $X$, where

$$E(x,y) = \inf_i \{1 - |\mu_i(x) - \mu_i(y)|\}$$
Necessity of scaling

Are there other similarity relations on the real numbers than \( E(x,y) = 1 - \min\{|x-y|,1\} \)?

Integration of scaling

A similarity relation is dependent on the measurement unit:

Celsius: \( E(20^\circ\text{C}, 20.5^\circ\text{C}) = 0.5 \)  Fahrenheit: \( E(68\text{F}, 68.9\text{F}) = 0.1 \)

scaling factor for Celsius/Fahrenheit: 1.8, since \( F = 18/10 \ C + 32 \)

\( E(x,y) = 1 - \min\{|c \cdot x - c \cdot y|, 1\} \)


\[ X = [a,b] \]

**Scaling**

c: \( X \rightarrow [0,\infty) \),

**Transformation**

\( f: X \rightarrow [0,\infty), \quad x \rightarrow \int_{a}^{x} c(t)dt \)

**Similarity relation**

\( E: X \times X \rightarrow [0,1], \quad (x,x') \rightarrow 1 - \min \{|f(x)-f(x')|, 1\} \)