9. Similarity Relations

Example 9.1

Specification of a partial control mapping ("good control actions")

gradient

		-40	-6	-3	0	3	6	40
	-70	22,5	15,0	15,0	10,0	10,0	5,0	5,0
	-50	22,5	15,0	10,0	10,0	5,0	5,0	0,0
deviation	-30	15,0	10,0	50,0	5,0	0,0	0,0	0,0
	0	5,0	5,0	0,0	0,0	0,0	-10,0	-15,0
	30	0,0	0,0	0,0	-5,0	-5,0	-10,0	-10,0
	50	0,0	-5,0	-5,0	-10,0	-15,0	-15,0	-22,5
	70	-5,0	-5,0	-15,0	-15,0	-15,0	-15,0	-15,0

current

Interpolation of this table

Additional knowledge was available: Some values are indistinguishable (from a measurement point of view) or should be treated in a similar way.

Problem 1: How to model such similarity information?

Problem 2: How to interpolate in the case of existing similarity information?

How to model similarity?

Proposal 1: equivalence relation

(I)	$x \approx x$	(reflexivity)
(II)	$x \approx y \leftrightarrow y \approx x$	(symmetry)
(III)	$x \approx y \wedge y \approx z \rightarrow x \approx z$	(transitivity)

- x and y similar ($x \approx y$) , if and only if $|x-y|<\varepsilon$, (ε fixed)

  $\approx$ is not transitive, Poincaré paradox

  $x \approx y$, $y \approx z$, $x \not\approx z$
- counterintuitive !

Proposal 2: similarity relation (multi-valued equivalence relation)

$[x \approx y]$ degree, to which $x \approx y$ holds

$E_{\approx}: X \times X \to [0,1], (x,y) \mapsto [x \approx y]$

(1) $E_{\approx}(x,x) = 1$

(2) $E_{\approx}(x,y) = E_{\approx}(y,x)$

(3) $\Pi(E_{\approx}(x,y), E_{\approx}(y,z)) \leq E_{\approx}(x,z)$, where Π is a *t*-norm

$E_{\approx}$ is called fuzzy equality relation, similarity relation, indistinguishability operator or tolerance relation.

- **Example**

  δ pseudo metric on X

  $\Pi(\alpha,\beta)=\max\{\alpha+\beta-1,0\}$ Lukasiewicz *t*-norm,

  Then

  $E_{\delta}(x,y) = 1-\min\{\delta(x,y), 1\}$ fuzzy equality relation

  $\delta_{E}(x,y) = 1-E(x,y)$ induced pseudo metric

  i.e. fuzzy equality and distance are dual notions

- **Formal Definition**

  $E: X\times X \to [0,1]$ similarity relation, iff

  (1) $E(x,x) = 1$

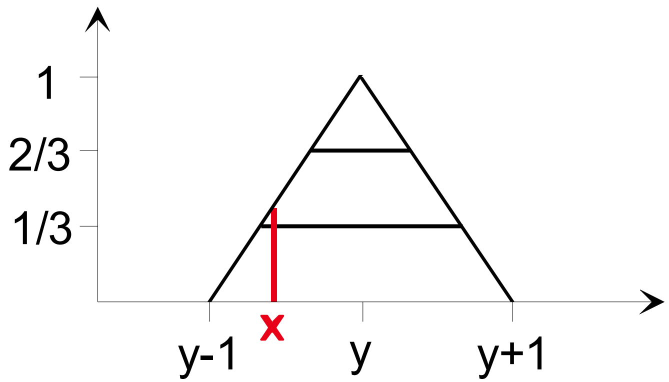
  (2) $E(x,y) = E(y,x)$

  (3) $\max\{(E(x,y)+ E(y,z)-1),0\} \leq E(x,z)$

Fuzzy Sets as a Derived Concept

$\delta(x,y) = |x-y|$ metric

$E_{\delta}(x,y) = 1 - \min\{|x-y|,1\}$ similarity relation



$\mu_y: X \rightarrow [0,1]$

$x \rightarrow E_{\delta}(x,y)$ fuzzy singleton

μ_y describes the “local“ similarities

Example 9.2

$$E : \Re x \Re \to [0,1], (x, y) \mapsto 1 - \min\{|x - y|, 1\}$$

is a similarity relation w.r.t the t-norm T_{Luka}

Def. 9.3

Let E be a similarity relation on X w.r.t. T .

$\mu \in F(X)$ is extensional iff T $(\mu(x), E(x, y)) \leq \mu(y)$ for all x,y.

Def. 9.4

Let E be a similarity relation on X w.r.t. T And $M \subseteq X$.

$$\mu_M : X \to [0,1], x \mapsto \sup\{E(x, x') \mid x' \in M\}$$

Is called extensional hull of M.

Example 9.5

A singleton is the extensional hull of $\{x_0\}$.

Def. 9.6

Let E, F be similarity relations on X and Y.

$\varsigma: X \to Y$ is extensional with respect to E, F iff $E(x,x') \le F(\varsigma(x), \varsigma(x'))$ holds.

Theorem 9.7

Let $E_1,\ldots,E_n$ similarity relations w.r.t. $\prod$ on $X_1,\ldots,X_n$.

Define $E:(X_1 \times \ldots \times X_n)^2 \to [0,1]$,

$$((x_1,\ldots,x_n),(y_1,\ldots,y_n)) \mapsto \min\{E_1(x_1,y_1),\ldots,E_n(x_n,y_n)\}$$

a) E is a similarity relation w.r.t. $\prod$ on $X_1 \times \ldots \times X_n$.

b) For all $i \in \{1,\ldots,n\}$ the projection π_i: $X_1 \times \ldots \times X_n \to X_i$ is extensional w.r.t. E and E_i.

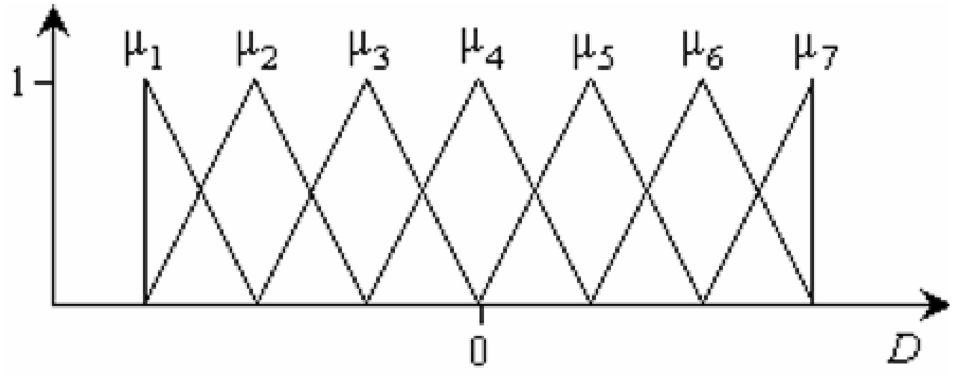
c) If E' is a similarity relation w.r.t. $\prod$ on $X_1 \times \ldots \times X_n$, i.e. all projections are extensional, then $E' \le E$ holds.

Remark 9.8

E is the biggest similarity relation for which all projections are extensional.

Specification of similarity relations

given a family of fuzzy sets that describes the “local“ similarities



then

- there is a similarity relation on D with induced fuzzy sets μ_i, iff

$$\sup_{x \in X}\{\mu_i(x)+\mu_j(x)-1\} \leq \inf_{y \in X}\{1-|\mu_i(y)-\mu_j(y)|\} \text{ for all } i, j$$

- if $\mu_i(x)+\mu_j(x) \leq 1$ for $i \neq j$ then there is a similarity relation E on X, where

$$E(x,y) = \inf_i \{1 - |\mu_i(x)- \mu_i(y)|\}$$

Necessity of scaling

Are there other similarity relations on the real numbers than

$E(x,y) = 1 - \min\{|x-y|,1\}$?

Integration of scaling

A similarity relation is dependent on the measurement unit:

Celsius: $E(20°C, 20.5°C) = 0.5$ Fahrenheit: $E(68F, 68.9F) = 0.1$

scaling factor for Celsius/Fahrenheit: 1.8, since F = 18/10 C + 32

$$E(x,y) = 1 - \min\{|c \cdot x - c \cdot y|, 1\}$$

$X = [a,b]$

Scaling

$c: X \to [0,\infty)$,

Transformation

$f: X \to [0,\infty), \quad x \to \int_a^x c(t)\,dt$

Similarity relation

$E: X \times X \to [0,1], (x,x') \to 1 - \min \{|f(x)-f(x')|, 1\}$