3. Fuzzy Operators

Standard Operators $(\mu \wedge \mu')(x) = \min\{\mu(x), \mu'(x)\}$

$(\mu \vee \mu')(x) = \max\{\mu(x), \mu'(x)\}$

$\neg\mu(x) = 1-\mu(x)$

Def. 3.1

Let μ be a fuzzy set. A fuzzy complement set $\tilde{\mu}$ of μ can be derived using a fuzzy negation operator ~

$\sim:[0,1]\to[0,1]$, $\tilde{\mu}(x) := \widetilde{\mu(x)}$

For ~ the following axioms hold:

c1) $\tilde{0} = 1,\ \tilde{1} = 0$ (extends crisp negation)

c2) $\forall_{a,b\in[0,1]}\ a \le b \Rightarrow \tilde{a} \ge \tilde{b}$ (monotony)

Furthermore it should satisfy:

c3) ~ is continuous

c4) $\forall_{a\in[0,1]}\ \tilde{\tilde{a}} = a$ (involution)

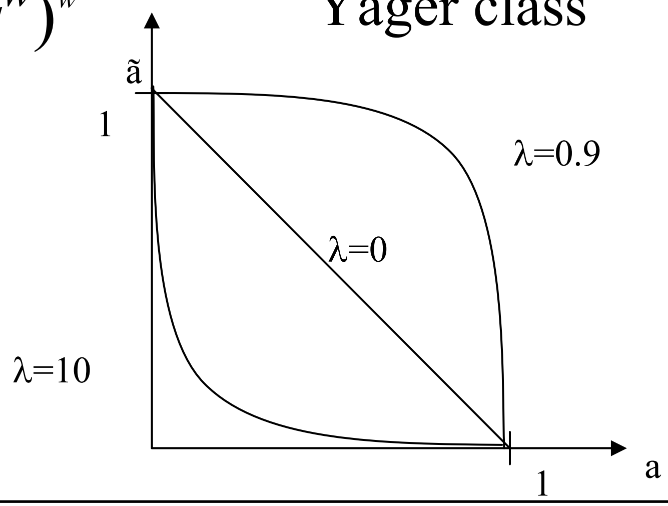
Examples 3.2

a) $\overline{}:[0,1]\rightarrow[0,1],\ x\rightarrow 1-x,\ \ \overline{\overline{x}} = x$

b) $\sim a = \begin{cases} b & \text{if} \quad a \le t \\ 0 & \text{if} \quad a > t \end{cases}$ $\qquad t\in[0,1)$, t is fixed (not continuous)

c) $\sim_\lambda (a) = \dfrac{1-a}{1+\lambda a}$ $\qquad$ Sugeno class $\quad \lambda\in(-1,\infty)$

d) $\sim_w (a) = (1-a^w)^{\frac{1}{w}}$ $\qquad$ Yager class $\quad w\in(0, \infty)$



Def. 3.3

A function $T:[0,1]^2 \to [0,1]$ is called a **t-norm** if the following holds:

i1) $\underset{a\in[0,1]}{\forall} T(a,1)=a$ (unit element)

i2) $\underset{a,b,c\in[0,1]}{\forall} a\le b \Rightarrow T(a,c)\le T(b,c)$ (monotony)

i3) $\underset{a,b\in[0,1]}{\forall} T(a,b)=T(b,a)$ (commutativity)

i4) $\underset{a,b,c\in[0,1]}{\forall} T(a,T(b,c))=T(T(a,b),c)$ (associativity)

It is evident that T is monotonic non-decreasing in both arguments and that $T(a,0)=T(0,a)\le T(0,1)=0$ holds.

Examples 3.4

a) $T_{\min}(a,b)=\min(a,b)$ standard fuzzy AND (Zadeh)

b) $T_{prod}(a,b)=a\cdot b$ algebraic product

c) $T_{Luka}(a,b)=\max(0,a+b-1)$ bounded difference

d) $T_{-1}(a,b)=\begin{cases} a & \text{if } b=1 \\ b & \text{if } a=1 \\ 0 & \text{otherwise} \end{cases}$ drastic intersection

Theorem 3.5

$T_{\min}(a,b)$ is the only idempotent t-norm, i.e. $T(a,a)=a.$

$$T_{-1}(a,b)\le T_{Luka}(a,b)\le T_{prod}(a,b)\le T_{\min}(a,b)$$

Def. 3.6

A function $\perp:[0,1]^2 \rightarrow [0,1]$ is called a **t-conorm** if, and only if:

v1) $\forall_{a\in[0,1]} \perp(a,0)=a$ (boundary)

v2) $\forall_{a,b,c\in[0,1]} a\leq b \Rightarrow \perp(a,c)\leq\perp(b,c)$ (monotony)

v3) $\forall_{a,b\in[0,1]} \perp(a,b)=\perp(b,a)$ (commutativity)

v4) $\forall_{a,b,c\in[0,1]} \perp(a,\perp(b,c))=\perp(\perp(a,b),c)$ (associativity)

Examples 3.7

$$\perp_{\max}(a,b)=\max(a,b)$$

$$\perp_{\text{sum}}(a,b)=a+b-ab$$

$$\perp_{\max}(a,b)=\max(a,b)$$

$$\perp_{-1}(a,b)=\begin{cases} a & \text{if } b=0 \\ b & \text{if } a=0 \\ 1 & \text{otherwise} \end{cases}$$

Def. 3.8

A function **t-norm** T / **t-conorm** ⊥ is called archimedean if it satisfies the following additional conditions

i5) T is continuous v5) ⊥ is continuous

i6) $\underset{a\in[0,1]}{\forall}\ \mathrm{T}(a,a)< a$ v6) $\underset{a\in[0,1]}{\forall}\ \perp(a,a)> a$ (subidempotency)

Theorem 3.9

A function $\mathrm{T}:[0,1]^2 \to[0,1]$ is an archimedean t-norm if there exists a monotonous decreasing generator function $\mathrm{f}:[0,1] \to[0,\infty)$
such that

$$\mathrm{f}(1)=0, \text{ and } \mathrm{T}(a,b)=\mathrm{f}^{(-1)}(\mathrm{f}(a)+\mathrm{f}(b))$$

$$\mathrm{f}^{(-1)}(y)=\begin{cases}\mathrm{f}^{-1}(y) & \text{if } y\in[0,\mathrm{f}(0)]\\ 0 & \text{if } y\in(\mathrm{f}(0),\infty)\end{cases}$$

Example 3.10

Yager-family, w>0

$f_w(x)=(1-x)^w$

$$T_{\text{Yager},w}(a,b) := 1 - \min\left(1, \left((1-a)^w + (1-b)^w\right)^{\frac{1}{w}}\right)$$

Def. 3.11

a) S-implication — based on $a \to b \Leftrightarrow \neg a \vee b$

$I(a,b)=T(\sim a,b)$ — $\sim$ strictly decreasing, involutive

e.g. $I_{Luka}(a,b)=\min(1,1-a+b)$

b) R-implication

$I(a,b)=1 \Leftrightarrow a \leq b$

$I(a,b)=\sup\{c \in [0,1] \mid T(a,c) \leq b$

e.g. $$I(a,b)=\begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$ using $T(a,b)=\min(a,b)$

c) QL-implication — from quantum logics

$a \to a \wedge b \Leftrightarrow \neg a \vee (a \wedge b)$

e.g. $I(a,b)=\max(1-a),\min(a,b))$ — Zadeh (using $T_{min}, \perp_{max}$, stand. $\sim$)