Semi-Graphoid and Graphoid Axioms

Unusually, one requires any notion of conditional independence to satisfy as a minimum the so-called Semi-Graphoid axioms: Let $W$, $X$, $Y$ and $Z$ be disjoint sets of attributes, with $W$, $X$ and $Y$ being non-empty. $X \perp\!\!\!\!\perp Y \mid Z$ shall denote “$X$ is conditionally independent of $Y$ given $Z$.”

- **Symmetry**: $X \perp\!\!\!\!\perp Y \mid Z \implies Y \perp\!\!\!\!\perp X \mid Z$
- **Decomposition**: $W \cup X \perp\!\!\!\!\perp Y \mid Z \implies X \perp\!\!\!\!\perp Y \mid Z$
- **Weak Union**: $W \cup X \perp\!\!\!\!\perp Y \mid Z \implies X \perp\!\!\!\!\perp Y \mid Z \cup W$
- **Contraction**: $(W \perp\!\!\!\!\perp X \mid Z) \land (W \perp\!\!\!\!\perp Y \mid Z \cup X)$
  \[ \implies W \perp\!\!\!\!\perp X \cup Y \mid Z \]

It is pleasant to also have the following axiom satisfied:

- **Intersection**: $(W \perp\!\!\!\!\perp X \mid Z \cup Y) \land (W \perp\!\!\!\!\perp Y \mid Z \cup X)$
  \[ \implies W \perp\!\!\!\!\perp X \cup Y \mid Z \]

All five axioms together are referred to as the Graphoid axioms. One can show that the conditional stochastic independence for strictly positive probability distributions satisfies the Graphoid axioms.

**Exercise 19**  

Semi-Graphoid and Graphoid Axioms

Show that the conditional stochastic (probabilistic) independence satisfies the decomposition axiom!

(Hint: In the probabilistic case $X \perp\!\!\!\!\perp Y \mid Z$ means that

\[ \forall x, y, z : \quad P(X = x, Y = y \mid Z = z) = P(X = x \mid Z = z) \cdot P(Y = y \mid Z = z) \]

or, equivalently, that

\[ \forall x, y, z : \quad P(X = x \mid Y = y, Z = z) = P(X = x \mid Z = z) . \]

The proof can be accomplished by inserting these relations and applying the well-known Kolmogorov axioms)

**Exercise 20**  

Semi-Graphoid and Graphoid Axioms

Show that the conditional stochastic (probabilistic) independence satisfies the weak union axiom!
Show that the conditional stochastic (probabilistic) independence does not satisfy the intersection axiom if we allow 0 probabilities!