Exercise Sheet 6

Semi-Graphoid and Graphoid Axioms

Unsually, one requires any notion of conditional independence to satisfy as a minimum the so-called *Semi-Graphoid axioms*: Let W, X, Y and Z be disjoint sets of attributes, with W, X and Y being non-empty. $X \perp\!\!\!\perp Y \mid Z$ shall denote „X is conditionally independent of Y given Z."

Symmetry	$X \perp\!\!\!\perp Y \mid Z \Longrightarrow Y \perp\!\!\!\perp X \mid Z$
Decomposition	$W \cup X \perp\!\!\!\perp Y \mid Z \Longrightarrow X \perp\!\!\!\perp Y \mid Z$
Weak Union	$W \cup X \perp\!\!\!\perp Y \mid Z \Longrightarrow X \perp\!\!\!\perp Y \mid Z \cup W$
Contraction	$(W \perp\!\!\!\perp X \mid Z) \wedge (W \perp\!\!\!\perp Y \mid Z \cup X) \Longrightarrow W \perp\!\!\!\perp X \cup Y \mid Z$

It is pleasant to also have the following axiom satisfied:

Intersection	$(W \perp\!\!\!\perp X \mid Z \cup Y) \wedge (W \perp\!\!\!\perp Y \mid Z \cup X) \Longrightarrow W \perp\!\!\!\perp X \cup Y \mid Z$

All five axioms together are referred to as the *Graphoid axioms*. One can show that the conditional stochastic independence for strictly positive probability distributions satisfies the Graphoid axioms.

Exercise 19 Semi-Graphoid and Graphoid Axioms

Show that the conditional stochastic (probabilistic) independence satisfies the decomposition axiom!

(Hint: In the probabilistic case $X \perp\!\!\!\perp Y \mid Z$ means that

$$\forall x, y, z: \quad P(X = x, Y = y \mid Z = z) = P(X = x \mid Z = z) \cdot P(Y = y \mid Z = z)$$

or, equivalently, that

$$\forall x, y, z: \quad P(X = x \mid Y = y, Z = z) = P(X = x \mid Z = z).$$

The proof can be accomplished by inserting these releations and applying the well-known Kolmogorov axioms)

Exercise 20 Semi-Graphoid and Graphoid Axioms

Show that the conditional stochastic (probabilistic) independence satisfies the weak union axiom!

Exercise 21 Semi-Graphoid and Graphoid Axioms

Show that the conditional stochastic (probabilistic) independence does **not** satisfy the intersection axiom if we allow 0 probabilities!