**Exercise Sheet 6**

### Semi-Graphoid and Graphoid Axioms

Unusually, one requires any notion of conditional independence to satisfy as a minimum the so-called *Semi-Graphoid axioms*: Let $W$, $X$, $Y$ and $Z$ be disjoint sets of attributes, with $W$, $X$ and $Y$ being non-empty. $X \perp \perp Y \mid Z$ shall denote „$X$ is conditionally independent of $Y$ given $Z$.“

- **Symmetry**: $X \perp \perp Y \mid Z \implies Y \perp \perp X \mid Z$
- **Decomposition**: $W \cup X \perp \perp Y \mid Z \implies X \perp \perp Y \mid Z$
- **Weak Union**: $W \cup X \perp \perp Y \mid Z \implies X \perp \perp Y \mid Z \cup W$
- **Contraction**: $(W \perp \perp X \mid Z) \wedge (W \perp \perp Y \mid Z \cup X) \implies W \perp \perp X \cup Y \mid Z$

It is pleasant to also have the following axiom satisfied:

- **Intersection**: $(W \perp \perp X \mid Z \cup Y) \wedge (W \perp \perp Y \mid Z \cup X) \implies W \perp \perp X \cup Y \mid Z$

All five axioms together are referred to as the *Graphoid axioms*. One can show that the conditional stochastic independence for strictly positive probability distributions satisfies the Graphoid axioms.

**Exercise 19**  
Semi-Graphoid and Graphoid Axioms

Show that the conditional stochastic (probabilistic) independence satisfies the decomposition axiom!

(Hint: In the probabilistic case $X \perp \perp Y \mid Z$ means that

$$
\forall x, y, z : \quad P(X = x, Y = y \mid Z = z) = P(X = x \mid Z = z) \cdot P(Y = y \mid Z = z)
$$

or, equivalently, that

$$
\forall x, y, z : \quad P(X = x \mid Y = y, Z = z) = P(X = x \mid Z = z).
$$

The proof can be accomplished by inserting these relations and applying the well-known Kolmogorov axioms)

**Exercise 20**  
Semi-Graphoid and Graphoid Axioms

Show that the conditional stochastic (probabilistic) independence satisfies the weak union axiom!
Exercise 21   Semi-Graphoid and Graphoid Axioms

Show that the conditional stochastic (probabilistic) independence does not satisfy the intersection axiom if we allow 0 probabilities!