Propagation in Belief Networks
Objective

- **Given:** Belief network \((V, E, P)\) with tree structure and \(P(V) > 0\). Set \(W \subseteq V\) of instantiated variables where a priori knowledge \(W \neq \emptyset\) is allowed

- **Desired:** \(P(B \mid W)\) for all \(B \in V\)

- **Notation:**
  - \(W_B^-\) subset of those variables of \(W\) that belong to the subtree of \((V, E)\) that has root \(B\)
  - \(W_B^+ = W \setminus W_B^-\)
  - \(s(B)\) set of direct successors of \(B\)
  - \(\Omega_B\) domain of \(B\)
  - \(b^*\) value that \(B\) is instantiated with
Example

\[ W = \{F, K, L, M\} \]

\[ W_B^+ = \{F, K\} \]

\[ s(B) = \{C, M, N\} \]

\[ W_B^- = \{L, M\} \]
Decomposition in the Tree

\[
P(B = b \mid W) = P(b \mid W_B^- \cup W_B^+) \quad \text{with } B \not\in W
\]

\[
= \frac{P(W_B^- \cup W_B^+ \cup \{b\})}{P(W_B^- \cup W_B^+)}
\]

\[
= \frac{P(W_B^- \cup W_B^+ \mid b)P(b)}{P(W_B^- \cup W_B^+)}
\]

\[
= \frac{P(W_B^- \mid b)P(W_B^+ \mid b)P(b)}{P(W_B^- \cup W_B^+)}
\]

\[
= \beta_{B,W} \begin{array}{l} P(W_B^- \mid b) \end{array} \begin{array}{l} P(b \mid W_B^+) \end{array} \quad \text{Evidence from “below” Evidence from “above”}
\]
\[\pi\text{- and } \lambda\text{-Values}\]

Since we ignore the constant \(\beta_{B,W}\) for the derivations below, the following designations are used instead of \(P(\cdot)\):

**\(\pi\)-values and \(\lambda\)-values**

Let \(B \in V\) be a variable and \(b \in \Omega_B\) a value of its domain. We define the \(\pi\)- and \(\lambda\)-values as follows:

\[
\lambda(b) = \begin{cases} 
P(W_B^- | b) & \text{if } B \notin W \\
1 & \text{if } B \in W \land b^* = b \\
0 & \text{if } B \in W \land b^* \neq b
\end{cases}
\]

\[
\pi(b) = P(b | W_B^+)
\]
\( \pi \)- and \( \lambda \)-Values

\[
\lambda(b) = \prod_{C \in s(B)} P(W_C^- \mid b) \quad \text{if } B \in W
\]

\[
\lambda(b) = 1 \quad \text{if } B \text{ leaf in } (V, E)
\]

\[
\pi(b) = P(b) \quad \text{if } B \text{ root in } (V, E)
\]

\[
P(b \mid W) = \alpha_{B,W} \cdot \lambda(b) \cdot \pi(b)
\]
\( \lambda \text{-Message} \)

**\( \lambda \)-message**

Let \( B \in V \) be an attribute and \( C \in s(B) \) its direct children with the respective domains \( \text{dom}(B) = \{B_1, \ldots, b_i, \ldots, b_k\} \) and \( \text{dom}(C) = \{c_1, \ldots, c_j, \ldots, c_m\} \).

\[
\lambda_{C \to B}(b_i) \overset{\text{Def}}{=} \sum_{j=1}^{m} P(c_j \mid b_i) \cdot \lambda(c_j), \quad i = 1, \ldots, k
\]

The vector

\[
\tilde{\lambda}_{C \to B} \overset{\text{Def}}{=} \left( \lambda_{C \to B}(b_i) \right)_{i=1}^{k}
\]

is called \( \lambda \)-message from \( C \) to \( B \).
Let $B \in V$ an attribute an $b \in \text{dom}(B)$ a value of its domain. Then

$$
\lambda(b) = \begin{cases} 
\rho_{B,W} \cdot \prod_{C \in s(B)} \lambda_C(b) & \text{if } B \notin W \\
1 & \text{if } B \in W \land b = b^* \\
0 & \text{if } B \in W \land b \neq b^*
\end{cases}
$$

with $\rho_{B,W}$ being a positive constant.
**π-Message**

**π-message**

Let $B \in V$ be a non-root node in $(V, E)$ and $A \in V$ its parent with domain $\text{dom}(A) = \{a_1, \ldots, a_j, \ldots, a_m\}$.

$$j = 1, \ldots, m :$$

$$\pi_{A \rightarrow B}(a_j) \overset{\text{Def}}{=} \begin{cases} 
\pi(a_j) \cdot \prod_{C \in s(A) \setminus \{B\}} \lambda_C(a_j) & \text{if } A \notin W \\
1 & \text{if } A \in W \land a = a^* \\
0 & \text{if } A \in W \land a \neq a^*
\end{cases}$$

The vector

$$\vec{\pi}_{A \rightarrow B} \overset{\text{Def}}{=} \left(\pi_{A \rightarrow B}(a_j)\right)_{j=1}^m$$

is called **π-message** from $A$ to $B$. 
Let $B \in V$ be a non-root node in $(V, E)$ and $A$ the parent node of $B$. Further let $b \in \text{dom}(B)$ be a value of $B$’s domain.

$$
\pi(b) = \mu_{B,W} \cdot \sum_{a \in \text{dom}(A)} P(b \mid a) \cdot \pi_{A \rightarrow B}(a)
$$

Let $A \notin W$ a non-instantiated attribute and $P(V) > 0$.

$$
\pi_{A \rightarrow B}(a_j) = \pi(a_j) \cdot \prod_{C \in s(A) \backslash \{B\}} \lambda_{C \rightarrow A}(a_j)
$$

$$
= \tau_{B,W} \cdot \frac{P(a_j \mid W)}{\lambda_{B \rightarrow A}(a_j)}
$$
Belief Tree:

Parameters:

\[ P(a_1) = 0.1 \quad P(b_1 \mid a_1) = 0.7 \]
\[ P(b_1 \mid a_2) = 0.2 \]
\[ P(d_1 \mid a_1) = 0.8 \quad P(c_1 \mid b_1) = 0.4 \]
\[ P(d_1 \mid a_2) = 0.4 \quad P(c_1 \mid b_2) = 0.001 \]

Desired:
\[ \forall X \in \{A, B, C, D\} : P(X \mid \emptyset) = ? \]
Propagation in Belief Trees (2)

Belief Tree:

Initialization Phase:

- Set all $\lambda$-messages and $\lambda$-values to 1.
Belief Tree:

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<tr>
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<th>( \lambda )</th>
<th>( P )</th>
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Initialization Phase:

- Set all \( \lambda \)-messages and \( \lambda \)-values to 1.
- \( \pi(a_1) = P(a_1) \) and \( \pi(a_2) = P(a_2) \)
Belief Tree:

Initialization Phase:

- Set all $\lambda$-messages and $\lambda$-values to 1.
- $\pi(a_1) = P(a_1)$ and $\pi(a_2) = P(a_2)$.
- $A$ sends $\pi$-messages to $B$ and $D$. 
Belief Tree:

Initialization Phase:

- Set all $\lambda$-messages and $\lambda$-values to 1.
- $\pi(a_1) = P(a_1)$ and $\pi(a_2) = P(a_2)$.
- $A$ sends $\pi$-messages to $B$ and $D$.
- $B$ and $D$ update their $\pi$-values.
Belief Tree:

Initialization Phase:

- Set all $\lambda$-messages and $\lambda$-values to 1.
- $\pi(a_1) = P(a_1)$ and $\pi(a_2) = P(a_2)$.
- $A$ sends $\pi$-messages to $B$ and $D$.
- $B$ and $D$ update their $\pi$-values.
- $B$ sends $\pi$-message to $C$. 

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Initialization Phase:

- Set all λ-messages and λ-values to 1.
- \( \pi(a_1) = P(a_1) \) and \( \pi(a_2) = P(a_2) \).
- A sends π-messages to B and D.
- B and D update their π-values.
- B sends π-message to C.
- C updates its π-value.
Belief Tree:

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<tr>
<td>a₂</td>
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<td>0.9</td>
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Initialization Phase:

- Set all λ-messages and λ-values to 1.
- \( \pi(a₁) = P(a₁) \) and \( \pi(a₂) = P(a₂) \).
- A sends \( \pi \)-messages to B and D.
- B and D update their \( \pi \)-values.
- B sends \( \pi \)-message to C.
- C updates its \( \pi \)-value.
- Initialization finished.
Larger Network (1): Parameters

\[
\begin{array}{c|c|c}
A & 0 \\
- & a_1 \ 0.4 \\
- & a_2 \ 0.6 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
P(B \mid A) & a_1 & a_2 \\
- & b_1 \ 0.2 & 0.3 \\
- & b_2 \ 0.8 & 0.7 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
P(C \mid A) & a_1 & a_2 \\
- & c_1 \ 0.1 & 0.25 \\
- & c_2 \ 0.9 & 0.75 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
P(F \mid C) & c_1 & c_2 \\
- & f_1 \ 0.3 & 0.6 \\
- & f_2 \ 0.7 & 0.4 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
P(D \mid B) & b_1 & b_2 \\
- & d_1 \ 0.5 & 0.35 \\
- & d_2 \ 0.5 & 0.65 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
P(E \mid B) & b_1 & b_2 \\
- & e_1 \ 0.15 & 0.45 \\
- & e_2 \ 0.85 & 0.55 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
P(G \mid C) & c_1 & c_2 \\
- & g_1 \ 0.25 & 0.1 \\
- & g_2 \ 0.75 & 0.9 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
P(H \mid F) & f_1 & f_2 \\
- & h_1 \ 0.65 & 0.2 \\
- & h_2 \ 0.35 & 0.8 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
P(I \mid F) & f_1 & f_2 \\
- & i_1 \ 0.25 & 0.5 \\
- & i_2 \ 0.75 & 0.5 \\
\end{array}
\]
Larger Network (2): After Initialization

\[
\begin{array}{cccc}
A & \pi & \lambda & P \\
\alpha_1 & 0.4 & 1 & 0.4 \\
\alpha_2 & 0.6 & 1 & 0.6 \\
\end{array}
\]

\[
\begin{array}{cccc}
B & \pi & \lambda & P \\
b_1 & 0.26 & 1 & 0.26 \\
b_2 & 0.74 & 1 & 0.74 \\
\end{array}
\]

\[
\begin{array}{cccc}
D & \pi & \lambda & P \\
d_1 & 0.389 & 1 & 0.389 \\
d_2 & 0.611 & 1 & 0.611 \\
\end{array}
\]

\[
\begin{array}{cccc}
E & \pi & \lambda & P \\
e_1 & 0.372 & 1 & 0.327 \\
e_2 & 0.628 & 1 & 0.628 \\
\end{array}
\]

\[
\begin{array}{cccc}
F & \pi & \lambda & P \\
f_1 & 0.543 & 1 & 0.543 \\
f_2 & 0.457 & 1 & 0.457 \\
\end{array}
\]

\[
\begin{array}{cccc}
G & \pi & \lambda & P \\
g_1 & 0.1285 & 1 & 0.1285 \\
g_2 & 0.8715 & 1 & 0.8715 \\
\end{array}
\]

\[
\begin{array}{cccc}
H & \pi & \lambda & P \\
h_1 & 0.4444 & 1 & 0.4444 \\
h_2 & 0.5556 & 1 & 0.5556 \\
\end{array}
\]

\[
\begin{array}{cccc}
I & \pi & \lambda & P \\
i_1 & 0.3643 & 1 & 0.3643 \\
i_2 & 0.6357 & 1 & 0.6357 \\
\end{array}
\]
Larger Network (3): Set Evidence $e_1, g_1, h_1$
Larger Network (4): Propagate Evidence

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<th>(\lambda)</th>
<th>(P)</th>
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<td>0.4</td>
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<tr>
<td>(a_2)</td>
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<td>0.6</td>
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| \(b_1\) | 0.26 | 1 | 0.26 |
| \(b_2\) | 0.74 | 1 | 0.74 |

| \(c_1\) | 0.19 | 1 | 0.19 |
| \(c_2\) | 0.81 | 1 | 0.81 |

| \(d_1\) | 0.389 | 1 | 0.389 |
| \(d_2\) | 0.611 | 1 | 0.611 |

| \(e_1\) | 1 | 1 |
| \(e_2\) | 0 | 0 |

| \(f_1\) | 0.543 | 1 | 0.543 |
| \(f_2\) | 0.457 | 1 | 0.457 |

| \(g_1\) | 1 | 1 |
| \(g_2\) | 0 | 0 |

| \(h_1\) | 1 | 1 |
| \(h_2\) | 0 | 0 |

| \(i_1\) | 0.3643 | 1 | 0.3643 |
| \(i_2\) | 0.6357 | 1 | 0.6357 |
Larger Network (5): Propagate Evidence, cont.

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Larger Network (7): Propagate Evidence, cont.
Larger Network (8): Propagate Evidence, cont.

\[
\begin{array}{cccc}
  \text{A} & \pi & \lambda & P \\
  a_1 & 0.4 & 0.39 & 0.4194 \\
  a_2 & 0.6 & 0.36 & 0.5806 \\
\end{array}
\]

\[
\begin{array}{cccc}
  \text{B} & \pi & \lambda & P \\
  b_1 & 0.26 & 0.15 & 0.1048 \\
  b_2 & 0.74 & 0.45 & 0.8952 \\
\end{array}
\]

\[
\begin{array}{cccc}
  \text{C} & \pi & \lambda & P \\
  c_1 & 0.19 & 0.0838 & 0.2948 \\
  c_2 & 0.81 & 0.047 & 0.7052 \\
\end{array}
\]

\[
\begin{array}{cccc}
  \text{D} & \pi & \lambda & P \\
  d_1 & 0.3657 & 1 & 0.3657 \\
  d_2 & 0.6343 & 1 & 0.6343 \\
\end{array}
\]

\[
\begin{array}{cccc}
  \text{E} & \pi & \lambda & P \\
  e_1 & 1 & 1 & \\
  e_2 & 0 & 0 & \\
\end{array}
\]

\[
\begin{array}{cccc}
  \text{F} & \pi & \lambda & P \\
  f_1 & 0.543 & 0.65 & 0.7943 \\
  f_2 & 0.457 & 0.2 & 0.2057 \\
\end{array}
\]

\[
\begin{array}{cccc}
  \text{G} & \pi & \lambda & P \\
  g_1 & 1 & 1 & \\
  g_2 & 0 & 0 & \\
\end{array}
\]

\[
\begin{array}{cccc}
  \text{H} & \pi & \lambda & P \\
  h_1 & 1 & 1 & \\
  h_2 & 0 & 0 & \\
\end{array}
\]

\[
\begin{array}{cccc}
  \text{I} & \pi & \lambda & P \\
  i_1 & 0.3014 & 1 & 0.3014 \\
  i_2 & 0.6986 & 1 & 0.6986 \\
\end{array}
\]
Larger Network (9): Propagate Evidence, cont.

\[
\begin{array}{c|c|c|c}
   A & \pi & \lambda & P \\
   \hline
   a_1 & 0.4 & 0.0198 & 0.3945 \\
   a_2 & 0.6 & 0.0202 & 0.0055 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
   B & \pi & \lambda & P \\
   \hline
   b_1 & 0.26 & 0.15 & 0.1048 \\
   b_2 & 0.74 & 0.45 & 0.8952 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
   C & \pi & \lambda & P \\
   \hline
   c_1 & 0.19 & 0.0838 & 0.2948 \\
   c_2 & 0.81 & 0.047 & 0.7052 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
   D & \pi & \lambda & P \\
   \hline
   d_1 & 0.3657 & 1 & 0.3657 \\
   d_2 & 0.6343 & 1 & 0.6343 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
   E & \pi & \lambda & P \\
   \hline
   e_1 & 1 & 1 & \\
   e_2 & 0 & 0 & \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
   F & \pi & \lambda & P \\
   \hline
   f_1 & 0.4891 & 0.65 & 0.7508 \\
   f_2 & 0.5109 & 0.2 & 0.2432 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
   G & \pi & \lambda & P \\
   \hline
   g_1 & 1 & 1 & \\
   g_2 & 0 & 0 & \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
   H & \pi & \lambda & P \\
   \hline
   h_1 & 1 & 1 & \\
   h_2 & 0 & 0 & \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
   I & \pi & \lambda & P \\
   \hline
   i_1 & 0.3014 & 1 & 0.3014 \\
   i_2 & 0.6986 & 1 & 0.6986 \\
\end{array}
\]
Larger Network (10): Propagate Evidence, cont.

\[
\begin{array}{cccc}
A & \pi & \lambda & P \\
a_1 & 0.4 & 0.0198 & 0.3945 \\
a_2 & 0.6 & 0.0202 & 0.6055 \\
\end{array}
\]

\[
\begin{array}{cccc}
B & \pi & \lambda & P \\
b_1 & 0.7077 & 0.15 & 0.1061 \\
b_2 & 1.9865 & 0.45 & 0.8939 \\
\end{array}
\]

\[
\begin{array}{cccc}
C & \pi & \lambda & P \\
c_1 & 0.1871 & 0.0838 & 0.2910 \\
c_2 & 0.8129 & 0.047 & 0.7090 \\
\end{array}
\]

\[
\begin{array}{cccc}
D & \pi & \lambda & P \\
d_1 & 0.3657 & 1 & 0.3657 \\
d_2 & 0.6343 & 1 & 0.6343 \\
\end{array}
\]

\[
\begin{array}{cccc}
E & \pi & \lambda & P \\
e_1 & 1 & 1 & \\
e_2 & 0 & 0 & \\
\end{array}
\]

\[
\begin{array}{cccc}
F & \pi & \lambda & P \\
f_1 & 0.4891 & 0.65 & 0.7568 \\
f_2 & 0.5109 & 0.2 & 0.2432 \\
\end{array}
\]

\[
\begin{array}{cccc}
G & \pi & \lambda & P \\
g_1 & 1 & 1 & \\
g_2 & 0 & 0 & \\
\end{array}
\]

\[
\begin{array}{cccc}
H & \pi & \lambda & P \\
h_1 & 1 & 1 & \\
h_2 & 0 & 0 & \\
\end{array}
\]

\[
\begin{array}{cccc}
I & \pi & \lambda & P \\
i_1 & 0.3108 & 1 & 0.3108 \\
i_2 & 0.6892 & 1 & 0.6892 \\
\end{array}
\]

\[
\begin{array}{cccc}
A & \pi & \lambda & P \\
(1.0115, 1.6819) & & & \\
(0.4194, 0.5806) & & & \\
\end{array}
\]
Larger Network (12): Propagate Evidence, cont.

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<th>λ</th>
<th>P</th>
</tr>
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<tbody>
<tr>
<td>g₁</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>g₂</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>π</th>
<th>λ</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>h₁</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>h₂</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>π</th>
<th>λ</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>i₁</td>
<td>0.3108</td>
<td>1</td>
<td>0.3108</td>
<td></td>
</tr>
<tr>
<td>i₂</td>
<td>0.6892</td>
<td>1</td>
<td>0.6892</td>
<td></td>
</tr>
</tbody>
</table>
Larger Network (15): Finished

A | \( \pi \) | \( \lambda \) | \( P \)  
---|---|---|---
\( a_1 \) | 0.4 | 0.0198 | 0.3945  
\( a_2 \) | 0.6 | 0.0202 | 0.6055  

B | \( \pi \) | \( \lambda \) | \( P \)  
---|---|---|---
\( b_1 \) | 0.7077 | 0.15 | 0.1061  
\( b_2 \) | 1.9865 | 0.45 | 0.8939  

C | \( \pi \) | \( \lambda \) | \( P \)  
---|---|---|---
\( c_1 \) | 0.1871 | 0.0838 | 0.2910  
\( c_2 \) | 0.8129 | 0.047 | 0.7090  

D | \( \pi \) | \( \lambda \) | \( P \)  
---|---|---|---
\( d_1 \) | 0.3659 | 1 | 0.3659  
\( d_2 \) | 0.6341 | 1 | 0.6341  

E | \( \pi \) | \( \lambda \) | \( P \)  
---|---|---|---
\( e_1 \) | 1 | 1 |  
\( e_2 \) | 0 | 0 |  

F | \( \pi \) | \( \lambda \) | \( P \)  
---|---|---|---
\( f_1 \) | 1.1657 | 0.65 | 0.7577  
\( f_2 \) | 1.2115 | 0.2 | 0.2423  

\( (0.7577, 0.2423) \)  

G | \( \pi \) | \( \lambda \) | \( P \)  
---|---|---|---
\( g_1 \) | 1 | 1 |  
\( g_2 \) | 0 | 0 |  

H | \( \pi \) | \( \lambda \) | \( P \)  
---|---|---|---
\( h_1 \) | 1 | 1 |  
\( h_2 \) | 0 | 0 |  

I | \( \pi \) | \( \lambda \) | \( P \)  
---|---|---|---
\( i_1 \) | 0.3106 | 1 | 0.3106  
\( i_2 \) | 0.6894 | 1 | 0.6894  

Rudolf Kruse, Matthias Steinbrecher, Pascal Held

Bayesian Networks
Propagation in Clique Trees
Problems

- The propagation algorithm as presented can only deal with trees.
- Can be extended to polytrees (i.e. singly connected graphs with multiple parents per node).
- However, it cannot handle networks that contain loops!
Main Objectives:

- Transform the cyclic directed graph into a secondary structure without cycles.
- Find a decomposition of the underlying joint distribution.

Task:

- Combine nodes of the original (primary) graph structure.
- These groups form the nodes of a secondary structure.
- Find a transformation that yields tree structure.
Secondary Structure:

- We will generate an undirected graph mimicking (some of) the conditional independence statements of the cyclic directed graph.
- Maximal cliques are identified and form the nodes of the secondary structure.
- Specify a so-called potential function for every clique such that the product of all potentials yields the initial joint distribution.
- In order to propagate evidence, create a **tree** from the clique nodes such that the following property is satisfied:
  
  If two cliques have some attributes in common, then these attributes have to be contained in every clique of the path connecting the two cliques. (called the **running intersection property**, RIP)

Justification:

- Tree: Unique path of evidence propagation.
- RIP: Update of an attribute reaches all cliques which contain it.
**Complete Graph**

An undirected Graph $G = (V, E)$ is called *complete*, if every pair of (distinct) nodes is connected by an edge.

**Induced Subgraph**

Let $G = (V, E)$ be an undirected graph and $W \subseteq V$ a selection of nodes. Then, $G_W = (W, E_W)$ is called the *subgraph of G induced by W* with $E_W$ being

$$E_W = \{(u, v) \in E \mid u, v \in W\}.$$
Complete Set, Clique

Let $G = (V, E)$ be an undirected graph. A set $W \subseteq V$ is called *complete* iff it induces a complete subgraph. It is further called a *clique*, iff $W$ is maximal, i.e. it is not possible to add a node to $W$ without violating the completeness condition.

a) $W$ is complete $\Leftrightarrow$ $W$ induces a complete subgraph

b) $W$ is a clique $\Leftrightarrow$ $W$ is complete and maximal

3 cliques

$C_1 = \{A, B, C, D\}$

$C_2 = \{B, D, E\}$

$C_3 = \{E, F\}$
Perfect Ordering

Let $G = (V, E)$ be an undirected graph with $n$ nodes and $\alpha = \langle v_1, \ldots, v_n \rangle$ a total ordering on $V$. Then, $\alpha$ is called **perfect**, if the following sets

$$\text{adj}(v_i) \cap \{v_1, \ldots, v_{i-1}\} \quad i = 1, \ldots, n$$

are complete, where $\text{adj}(v_i) = \{w \mid (v_i, w) \in E\}$ returns the adjacent nodes of $v_i$.

$\alpha = \langle A, C, D, F, E, B, H, G \rangle$

$\alpha$ is a perfect ordering
Prerequisites (4)

Running Intersection Property

Let $G = (V, E)$ be an undirected graph with $p$ cliques. An ordering of these cliques has the running intersection property (RIP), if for every $j > 1$ there exists an $i < j$ such that:

$$C_j \cap \left( C_1 \cup \cdots \cup C_{j-1} \right) \subseteq C_i$$

Let $\xi = \langle C_1, C_2, C_3, C_4, C_5, C_6 \rangle$

$$\begin{array}{c|c|c}
 j & C_j \cap \left( C_1 \cup \cdots \cup C_{j-1} \right) & i \\
 2 & \{C\} & \subseteq C_1 \\
 3 & \{D, F\} & \subseteq C_2 \\
 4 & \{D, E\} & \subseteq C_3 \\
 5 & \{E, F\} & \subseteq C_3 \\
 6 & \{F\} & \subseteq C_5 \\
\end{array}$$

$\xi$ has running intersection property
If a node ordering $\alpha$ of an undirected graph $G = (V, E)$ is perfect and the cliques of $G$ are ordered according to the highest rank (w.r.t. $\alpha$) of the containing nodes, then this clique ordering has RIP.

<table>
<thead>
<tr>
<th>Clique</th>
<th>Rank</th>
<th>$\rightarrow$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${A, C}$</td>
<td>$\max{\alpha(A), \alpha(C)}$</td>
<td>$2 \rightarrow C_1$</td>
</tr>
<tr>
<td>${C, D, F}$</td>
<td>$\max{\alpha(C), \alpha(D), \alpha(F)}$</td>
<td>$4 \rightarrow C_2$</td>
</tr>
<tr>
<td>${D, E, F}$</td>
<td>$\max{\alpha(D), \alpha(E), \alpha(F)}$</td>
<td>$5 \rightarrow C_3$</td>
</tr>
<tr>
<td>${B, D, E}$</td>
<td>$\max{\alpha(B), \alpha(D), \alpha(E)}$</td>
<td>$6 \rightarrow C_4$</td>
</tr>
<tr>
<td>${F, E, H}$</td>
<td>$\max{\alpha(F), \alpha(E), \alpha(H)}$</td>
<td>$7 \rightarrow C_5$</td>
</tr>
<tr>
<td>${F, G}$</td>
<td>$\max{\alpha(F), \alpha(G)}$</td>
<td>$8 \rightarrow C_6$</td>
</tr>
</tbody>
</table>

How to get a perfect ordering?
Triangulated Graph

An undirected graph is called *triangulated* if every simple loop (i.e. path with identical start and end node but with any other node occurring at most once) of length greater than 3 has a chord.
Maximum Cardinality Search

Let $G = (V, E)$ be an undirected graph. An ordering according maximum cardinality search (MCS) is obtained by first assigning 1 to an arbitrary node. If $n$ numbers are assigned the node that is connected to most of the nodes already numbered gets assigned number $n + 1$.

3 can be assigned to $D$ or $F$
6 can be assigned to $H$ or $B$
An undirected graph is triangulated iff the ordering obtained by MCS is perfect.

To check whether a graph is triangulated is efficient to implement. The optimization problem that is related to the triangulation task is NP-hard. However, there are good heuristics.

**Moral Graph** (Repetition)

Let $G = (V, E)$ be a directed acyclic graph. If $u, w \in W$ are parents of $v \in V$ connect $u$ and $w$ with an (arbitrarily oriented) edge. After the removal of all edge directions the resulting graph $G_m = (V, E')$ is called the *moral graph* of $G$. 
Given directed graph.
Join-Tree Construction (2)

• Moral graph
Join-Tree Construction (3)

- Moral graph
- Triangulated graph
Join-Tree Construction (4)

- Moral graph
- Triangulated graph
- MCS yields perfect ordering
Join-Tree Construction (5)

- Moral graph
- Triangulated graph
- MCS yields perfect ordering
- Clique order has RIP
Join-Tree Construction (6)

- Moral graph
- Triangulated graph
- MCS yields perfect ordering
- Clique order has RIP
- Form a join-tree

Two cliques can be connected if they have a non-empty intersection. The generation of the tree follows the RIP. In case of a tie, connect cliques with the largest intersection. (e.g. $DBE—FED$ instead of $DBE—CFD$) Break remaining ties arbitrarily.
Example: Expert Knowledge

- **Qualitative knowledge:**
  Metastatic cancer is a possible cause of brain tumor, and is also an explanation for increased total serum calcium. In turn, either of these could explain a patient falling into a coma. Severe headache is also possibly associated with a brain tumor.

- **Special case:**
  The patient has heavy headache.

- **Query:**
  Will the patient fall into coma?
**Example: Choice of State Space**

<table>
<thead>
<tr>
<th>Attribute</th>
<th>Possible Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$ metastatic cancer</td>
<td>$\text{dom}(A) = {a_1, a_2}$ $\cdot_1 = \text{existing}$</td>
</tr>
<tr>
<td>$B$ increased total serum calcium</td>
<td>$\text{dom}(B) = {b_1, b_2}$ $\cdot_2 = \text{notexisting}$</td>
</tr>
<tr>
<td>$C$ brain tumor</td>
<td>$\text{dom}(C) = {c_1, c_2}$</td>
</tr>
<tr>
<td>$D$ coma</td>
<td>$\text{dom}(D) = {d_1, d_2}$</td>
</tr>
<tr>
<td>$E$ severe headache</td>
<td>$\text{dom}(E) = {e_1, e_2}$</td>
</tr>
</tbody>
</table>

Exhaustive state space:

$$\Omega = \text{dom}(A) \times \text{dom}(B) \times \text{dom}(C) \times \text{dom}(D) \times \text{dom}(E)$$

Marginal and conditional probabilities have to be specified!
Example: Qualitative Knowledge

\[
\begin{align*}
P(e_1 | c_1) &= 0.8 \\
P(e_1 | c_2) &= 0.6
\end{align*}
\]

\{ headaches common, but more common if tumor present \}

\[
\begin{align*}
P(d_1 | b_1, c_1) &= 0.8 \\
P(d_1 | b_1, c_2) &= 0.8 \\
P(d_1 | b_2, c_1) &= 0.8 \\
P(d_1 | b_2, c_2) &= 0.05
\end{align*}
\]

\{ coma rare but common, if either cause is present \}

\[
\begin{align*}
P(b_1 | a_1) &= 0.8 \\
P(b_1 | a_2) &= 0.2
\end{align*}
\]

\{ increased calcium uncommon, but common consequence of metastases \}

\[
\begin{align*}
P(c_1 | a_1) &= 0.2 \\
P(c_1 | a_2) &= 0.05
\end{align*}
\]

\{ brain tumor rare, and uncommon consequence of metastases \}

\[
P(a_1) = 0.2
\]

\{ incidence of metastatic cancer in relevant clinic \}
Example: Metastatic Cancer

Dependencies

Moralization/Triangulation

MCS, hyper graph

Clique tree with separator sets
Quantitative knowledge:

<table>
<thead>
<tr>
<th>(a, b, c)</th>
<th>P(a, b, c)</th>
<th>(b, c, d)</th>
<th>P(b, c, d)</th>
<th>(c, e)</th>
<th>P(c, e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a₁, b₁, c₁</td>
<td>0.032</td>
<td>b₁, c₁, d₁</td>
<td>0.032</td>
<td>c₁, e₁</td>
<td>0.064</td>
</tr>
<tr>
<td>a₂, b₁, c₁</td>
<td>0.008</td>
<td>b₂, c₁, d₁</td>
<td>0.032</td>
<td>c₂, e₁</td>
<td>0.552</td>
</tr>
<tr>
<td>···</td>
<td>···</td>
<td>···</td>
<td>···</td>
<td>···</td>
<td>···</td>
</tr>
<tr>
<td>a₂, b₂, c₂</td>
<td>0.608</td>
<td>b₂, c₂, d₂</td>
<td>0.608</td>
<td>c₁, e₂</td>
<td>0.016</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>c₂, e₂</td>
<td>0.368</td>
</tr>
</tbody>
</table>

Potential representation:

\[
P(A, B, C, D, E, ) = P(A \mid \emptyset)P(B \mid A)P(C \mid A)P(D \mid BC)P(E \mid C) = \frac{P(A, B, C)P(B, C, D), P(C, E)}{P(BC)P(C)}
\]
Propagation on Cliques (4)

Propagation:

- \( P(d_1) = 0.32 \), evidence \( E = e_1 \), desired: \( P^*(\ldots) = P(\cdot \mid \{e_1\}) \)

\[
\begin{align*}
P^*(c) &= P(c \mid e_1) \quad \text{conditional marginal distribution} \\
P^*(b, c, d) &= \frac{P(b, c, d)}{P(c)} P(c \mid e_1) \quad \text{multipl./division with separation prob.} \\
P^*(b, c) &= P(b, c \mid e_1) \quad \text{calculate marginal distributions} \\
P^*(a, b, c) &= \frac{P(a, b, c)}{P(b, c)} P(b, c \mid e_1) \quad \text{multipl./division with separation prob.} \\
P^*(d_1) &= P(d_1 \mid e_1) = 0.33
\end{align*}
\]
Marginal distributions in the HUGIN tool.
Conditional marginal distributions with evidence $E = e_1$
Potential Representation

Let $V = \{X_j\}$ be a set of random variables $X_j : \Omega \rightarrow \text{dom}(X_j)$ and $P$ the joint distribution over $V$. Further, let

$$\{W_i \mid W_i \subseteq V, 1 \leq i \leq p\}$$

a family of subsets of $V$ with associated functions

$$\psi_i : \bigtimes_{X_j \in W_i} \text{dom}(X_j) \rightarrow \mathbb{R}$$

It is said that $P(V)$ factorizes according $(\{W_1, \ldots, W_p\}, \{\psi_1, \ldots, \psi_p\})$ if $P(V)$ can be written as:

$$P(v) = k \cdot \prod_{i=1}^{p} \psi_i(w_i)$$

where $k \in \mathbb{R}$, $w_i$ is a realization of $W_i$ that meets the values of $v$. 
Example

\[ V = \{A, B, C\}, \; W_1 = \{A, B\}, \; W_2 = \{B, C\} \]

\[ \text{dom}(A) = \{a_1, a_2\} \]
\[ \text{dom}(B) = \{b_1, b_2\} \]
\[ \text{dom}(C) = \{c_1, c_2\} \]

\[ P(a, b, c) = \frac{1}{8} \]

\[ \psi_1 : \{a_1, a_2\} \times \{b_1, b_2\} \rightarrow \mathbb{R} \]
\[ \psi_2 : \{b_1, b_2\} \times \{c_1, c_2\} \rightarrow \mathbb{R} \]

\[ \psi_1(a, b) = \frac{1}{4} \]
\[ \psi_2(b, c) = \frac{1}{2} \]

\( (\{W_1, W_2\}, \{\psi_1, \psi_2\}) \) is a potential representation of \( P \).
Factorization of a Belief Network

Let \((V, E, P)\) be an belief network and \(\{C_1, \ldots, C_p\}\) the cliques of the join tree. For every node \(v \in V\) choose a clique \(C\) such that \(v\) and all of its parents are contained in \(C\), i.e. \(\{v\} \cup c(v) \subseteq C\). The chosen clique is designated as \(f(v)\).

To arrive at a factorization \((\{C_1, \ldots, C_p\}, \{\psi_1, \ldots, \psi_p\})\) of \(P\) the factor potentials are:

\[
\psi_i(c_i) = \prod_{v : f(v) = C_i} P(v \mid c(v))
\]

Separator Sets and Residual Sets

Let \(\{C_1, \ldots, C_p\}\) be a set of cliques w. r. t. \(V\). The sets

\[
S_i = C_i \cap (C_1 \cup \cdots \cup C_{i-1}), \quad i = 1, \ldots, p, \quad S_1 = \emptyset
\]

are called separator sets with their corresponding residual sets

\[
R_i = C_i \backslash S_i
\]
Decomposition w. r. t. a Join-Tree

- Given a clique ordering \( \{C_1, \ldots, C_p\} \) that satisfies the RIP, we can easily conclude the following separation statements:

\[
R_i \perp \perp (C_1 \cup \cdots \cup C_{i-1}) \setminus S_i \mid S_i \quad \text{for } i > 1
\]

- Hence, we can formulate the following factorization:

\[
P(X_1, \ldots, X_n) = \prod_{i=1}^{p} P(R_i \mid S_i),
\]

which also gives us a representation in terms of conditional probabilities (as for directed graphs before).
Example

\[
\begin{align*}
S_1 &= \emptyset & R_1 &= \{A, B, C\} & f(A) &= C_1 \\
S_2 &= \{B, C\} & R_2 &= \{D\} & f(B) &= C_1 \\
S_3 &= \{C\} & R_3 &= \{E\} & f(C) &= C_1 \\
& & f(D) &= C_2 \\
& & f(E) &= C_3
\end{align*}
\]

\[
\begin{align*}
\psi_1(C_1) &= P(A, B, C \mid \emptyset) = P(A) \cdot P(C \mid A) \cdot P(B \mid A) \\
\psi_2(C_2) &= P(D \mid B, C) \\
\psi_3(C_3) &= P(E \mid C)
\end{align*}
\]

Propagation is accomplished by sending messages across the cliques in the tree. The emerging potentials are maintained by each clique.
Main Idea

- Incorporate evidence into the clique potentials.
- Since we are dealing with a tree structure, exploit the fact that a clique “separates” all its neighboring cliques (and their respective subtrees) from each other.
- Apply a message passing scheme to inform neighboring cliques about evidence.
- Since we do not have edge directions, we will only need one type of message.
- After having updated all cliques’ potentials, we marginalize (and normalize) to get the probabilities of single attributes.
• Every clique $C_i$ maintains a potential function $\psi_i$.

• If for an attribute $E$ some evidence $e$ becomes known, we alter all potential functions of cliques containing $E$ as follows:

$$
\psi_{i}^*(c_i) = \begin{cases} 
0, & \text{if a value in } c_i \text{ is inconsistent with } e \\
\psi_i(c_i), & \text{otherwise}
\end{cases}
$$

• All other potential functions are unchanged.
In general:

- Clique $C_i$ has $q$ neighboring cliques $B_1, \ldots, B_q$.
- $C_{ij}$ is the set of cliques in the subtree containing $C_i$ after dropping the link to $B_j$.
- $X_{ij}$ is the set of attributes in the cliques of $C_{ij}$.
- $V = X_{ij} \cup X_{ji}$ (complementary sets)
- $S_{ij} = S_{ji} = C_i \cap C_j$ (not shown here)
- $R_{ij} = X_{ij} \setminus S_{ij}$ (not shown here)

Here:

- Neighbors of $C_1$: $\{C_2, C_4, C_3\}$, $C_{13} = \{C_1, C_2, C_4\}$
- $X_{13} = \{A, B, C, D, E, G\}$, $S_{13} = \{C, G\}$
- $V = X_{13} \cup X_{31} = \{A, B, C, D, E, F, G, H\}$
- $R_{13} = \{A, B, D, E\}$, $R_{31} = \{F, H\}$
Task: Calculate $P(s_{ij})$:

\[
V \setminus S_{ij} = (X_{ij} \cup X_{ji}) \setminus S_{ij} = (X_{ij} \setminus S_{ij}) \cup (X_{ji} \setminus S_{ij}) = R_{ij} \cup R_{ji}
\]

\[
V \setminus S_{13} = (X_{13} \cup X_{31}) \setminus S_{13} = R_{13} \cup R_{31}
\]

\[
V \setminus \{C, G\} = \{A, B, D, E\} \cup \{F, H\} = \{A, B, D, E, F, H\}
\]

Note: $R_{ij}$ is the set of attributes that are in $C_i$’s subtree but not in $B_j$’s. Therefore, $R_{ij}$ and $R_{ji}$ are always disjoint.
Task: Calculate $P(s_{ij})$:

$$
P(s_{ij}) = \sum_{v \setminus s_{ij}} \prod_{k=1}^{m} \psi_k(c_k)$$

Last slide:

$$\sum_{r_{ij} \cup r_{ji}} \prod_{k=1}^{m} \psi_k(c_k)$$

Sum rule:

$$\left( \sum_{r_{ij}} \prod_{c_k \in C_{ij}} \psi_k(c_k) \right) \cdot \left( \sum_{r_{ji}} \prod_{c_k \in C_{ji}} \psi_k(c_k) \right)$$

$$= M_{ij}(s_{ij}) \cdot M_{ji}(s_{ij})$$

$M_{ij}$ is the message sent from $C_i$ to neighbor $B_j$ and vice versa.
Task: Calculate $P(c_i)$:

$$V \setminus C_i = \left( \bigcup_{k=1}^{q} X_{ki} \right) \setminus C_i$$

$$= \bigcup_{k=1}^{q} \left( X_{ki} \setminus C_i \right)$$

$$= \bigcup_{k=1}^{q} R_{ki}$$

Example:

$$V \setminus C_1 = R_{21} \cup R_{41} \cup R_{31}$$

$$\{A, D, F, H\} = \{A\} \cup \{D\} \cup \{F, H\}$$
**Task:** Calculate $P(c_i)$:

$$
P(c_i) = \sum_{v \setminus c_i} \prod_{j=1}^{m} \psi_j(c_j)$$

**Marginalization Decomposition**

$$= \psi_i(c_i) \sum_{v \setminus c_i \ i \neq j} \prod \psi_j(c_j)$$

$$= \psi_i(c_i) \sum_{r_{1i} \cup \ldots \cup r_{qi} \ i \neq j} \prod \psi_j(c_j)$$

$$= \psi_i(c_i) \left( \sum_{r_{1i}} \prod_{c_k \in C_{1i}} \psi_k(c_k) \right) \cdot \left( \sum_{r_{qi}} \prod_{c_k \in C_{qi}} \psi_k(c_k) \right)$$

$$= \psi_i(c_i) \prod_{j=1}^{q} M_{ji}(s_{ij})$$
Example: $P(c_1)$:

$$P(c_1) = \psi_1(c_1) M_{21}(s_{12}) M_{41}(s_{14}) M_{31}(s_{13})$$

$M_{ij}(s_{ij})$ can be simplified further (without proof):

$$M_{ij}(s_{ij}) = \sum_{r_{ij}} \prod_{c_k \in C_{ij}} \psi_k(c_k)$$

$$= \sum_{c_i \setminus s_{ij}} \psi_i(c_i) \prod_{k \neq j} M_{ki}(s_{ki})$$
Final Algorithm

- **Input:** Join tree \((C, \Psi)\) over set of variables \(V\) and evidence \(E = e\).
- **Output:** The a-posteriori probability \(P(x_i | e)\) for every non-evidential \(X_i\).
- **Initialization:** Incorporate evidence \(E = e\) into potential functions.
- **Iterations:**
  1. For every clique \(C_i\) do: For every neighbor \(B_j\) of \(C_i\) do: If \(C_i\) has received all messages from the other neighbors, calculate and send \(M_{ij}(s_{ij})\) to \(B_j\).
  2. Repeat step 1 until no message is calculated.
  3. Calculate the joint probability distribution for every clique:
     \[
     P(c_i) \propto \psi_i(c_i) \prod_{j=1}^{q} M_{ji}(s_{ij})
     \]
  4. For every \(X \in V\) calculate the a-posteriori probability:
     \[
     P(x_i | e) = \sum_{c_k \setminus x_i} P(c_k)
     \]
     where \(C_k\) is the smallest clique containing \(X_i\).
**Goals:** Find the marginal distributions and update them when evidence $H = h_1$ becomes known.

**Steps:**
1. Transform network into join-tree.
2. Specify factor potentials.
3. Propagate “zero” evidence to obtain the marginals before evidence is present.
4. Update factor potentials w.r.t. the evidence and do another propagation run.
Example: Step 1: Find a Join-Tree

Join-Tree creation:
Example: Step 1: Find a Join-Tree

Join-Tree creation:
1. Moralize the graph.
Example: Step 1: Find a Join-Tree

Join-Tree creation:
1. Moralize the graph.
2. Not yet triangulated.
Example: Step 1: Find a Join-Tree

Join-Tree creation:
1. Moralize the graph.
2. Triangulate the graph.
Example: Step 1: Find a Join-Tree

Join-Tree creation:
1. Moralize the graph.
2. Triangulate the graph.
3. Identify the maximal cliques.
Example: Step 1: Find a Join-Tree

Example Bayesian network

One of the join trees
Example: Step 2: Specify the Factor Potentials

Decomposition of $P(A, B, C, D, E, F, G, H)$:

$$P(a, b, c, d, e, f, g, h) = \prod_{i=1}^{5} \Psi_i(c_i)$$

$$= \Psi_1(b, c, e, g) \cdot \Psi_2(a, b, c) \cdot \Psi_3(c, f, g) \cdot \Psi_4(b, d) \cdot \Psi_5(g, f, h)$$

Where to get the factor potentials from?
Example: Step 2: Specify the Factor Potentials

As long as the factor potentials multiply together as on the previous slide, we are free to choose them.

- **Option 1:** A factor potential of clique $C_i$ is the product of all conditional probabilities of all node families properly contained in $C_i$:

  \[
  \Psi_i(c_i) = 1 \cdot \prod_{\{X_i\} \cup Y_i \subseteq C_i \land \text{parents}(X_i) = Y_i} P(x_i \mid y_i)
  \]

  The 1 stresses that if no node family satisfies the product condition, we assign a constant 1 to the potential.

- **Option 2:** Choose potentials from the decomposition formula:

  \[
  P(\bigcup_{i=1}^{n} C_i) = \prod_{i=1}^{n} P(C_i) \prod_{j=1}^{m} P(S_j)
  \]
Example: Step 2: Specify the Factor Potentials

- **Option 1:** Factor potentials according to the conditional distributions of the node families of the underlying Bayesian network:

\[
\begin{align*}
\Psi_1(b, c, e, g) &= P(e | b, c) \cdot P(g | e, b) \\
\Psi_2(a, b, c) &= P(b | a) \cdot P(c | a) \cdot P(a) \\
\Psi_3(c, f, g) &= P(f | c) \\
\Psi_4(b, d) &= P(d | b) \\
\Psi_5(g, f, h) &= P(h | g, f)
\end{align*}
\]

(This assignment of factor potentials is used in this example.)

- **Option 2:** Factor potentials chosen from the join-tree decomposition:

\[
\begin{align*}
\Psi_1(b, c, e, g) &= P(b, e | c, g) \\
\Psi_2(a, b, c) &= P(a | b, c) \\
\Psi_3(c, f, g) &= P(c | f, g) \\
\Psi_4(b, d) &= P(d | b) \\
\Psi_5(g, f, h) &= P(h, g, f)
\end{align*}
\]
Example: Closer Look on Option 2: Separation in a Join-Tree

Encoded independence statements:

Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.
Encoded independence statements:

Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.

\[ A \perp D, E, F, G, H \mid B, C \]
Example: Closer Look on Option 2: Separation in a Join-Tree

Encoded independence statements:

Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.

\[ A \perp D, E, F, G, H \mid B, C \]
\[ D \perp A, C, E, F, G, H \mid B \]
Example: Closer Look on Option 2: Separation in a Join-Tree

Encoded independence statements:

Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.

\[ A \perp D, E, F, G, H \mid B, C \]
\[ D \perp A, C, E, F, G, H \mid B \]
\[ A, B, E, D \perp F, H \mid G, C \]
Encoded independence statements:

Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.

\[
\begin{align*}
A & \perp D, E, F, G, H \mid B, C \\
D & \perp A, C, E, F, G, H \mid B \\
A, B, E, D & \perp F, H \mid G, C \\
H & \perp A, B, C, D, E \mid F, G
\end{align*}
\]
The four separation statements translate into the following independence statements:

\[ A \perp \perp D, E, F, G, H \mid B, C \iff P(A \mid B, C, D, E, F, G, H) = P(A \mid B, C) \]
\[ D \perp \perp A, C, E, F, G, H \mid B \Rightarrow P(D \mid B, C, E, F, G, H) = P(D \mid B) \]
\[ A, B, E, D \perp \perp F, H \mid G, C \Rightarrow P(B, E \mid G, C, F, H) = P(B, E \mid G, C) \]
\[ H \perp \perp A, B, C, D, E \mid F, G \Rightarrow P(C \mid F, G, H) = P(C \mid F, G) \]

According to the chain rule we always have the following relation:

\[
\]
The four separation statements translate into the following independence statements:

\[
\begin{align*}
A \perp D, E, F, G, H \mid B, C & \iff P(A \mid B, C, D, E, F, G, H) = P(A \mid B, C) \\
D \perp A, C, E, F, G, H \mid B & \Rightarrow P(D \mid B, C, E, F, G, H) = P(D \mid B) \\
A, B, E, D \perp F, H \mid G, C & \Rightarrow P(B, E \mid G, C, F, H) = P(B, E \mid G, C) \\
H \perp A, B, C, D, E \mid F, G & \Rightarrow P(C \mid F, G, H) = P(C \mid F, G)
\end{align*}
\]

Exploiting the above independencies yields:

\[
P(A, B, C, D, E, F, G, H) = P(A \mid B, C) \cdot P(D \mid B) \cdot P(B, E \mid C, G) \cdot P(C \mid F, G) \cdot P(F, G, H)
\]
The four separation statements translate into the following independence statements:

\[
\begin{align*}
A & \indep D, E, F, G, H \mid B, C \Leftrightarrow P(A \mid B, C, D, E, F, G, H) = P(A \mid B, C) \\
D & \indep A, C, E, F, G, H \mid B \Rightarrow P(D \mid B, C, E, F, G, H) = P(D \mid B) \\
A, B, E, D & \indep F, H \mid G, C \Rightarrow P(B, E \mid G, C, F, H) = P(B, E \mid G, C) \\
H & \indep A, B, C, D, E \mid F, G \Rightarrow P(C \mid F, G, H) = P(C \mid F, G)
\end{align*}
\]

Getting rid of the conditions results in the final decomposition equation:

\[
P(A, B, C, D, E, F, G, H) = P(A \mid B, C)P(D \mid B)P(B, E \mid C, G)P(C \mid F, G)P(F, G, H)
\]

\[
= \frac{P(A, B, C)P(D, B)P(B, E, C, G)P(C, F, G)P(F, G, H)}{P(B, C)P(B)P(C, G)P(F, G)}
\]

\[
= \frac{P(C_1)P(C_2)P(C_3)P(C_4)P(C_5)}{P(S_{12})P(S_{14})P(S_{13})P(S_{35})}
\]
According to the join-tree propagation algorithm, the probability distributions of all clique instantiations \( c_i \) is calculated as follows:

\[
P(c_i) \propto \Psi_i(c_i) \prod_{j=1}^{q} M_{ji}(s_{ij})
\]

Spelt out for our example, we get:

\[
P(c_1) = P(b, c, e, g) = \Psi_1(b, c, e, g) \cdot M_{21}(b, c) \cdot M_{31}(c, g) \cdot M_{41}(b)
\]
\[
P(c_2) = P(a, b, c) \propto \Psi_2(a, b, c) \cdot M_{12}(b, c)
\]
\[
P(c_3) = P(c, f, g) \propto \Psi_3(c, f, g) \cdot M_{13}(c, g) \cdot M_{53}(f, g)
\]
\[
P(c_4) = P(b, d) \propto \Psi_4(b, d) \cdot M_{14}(b)
\]
\[
P(c_5) = P(f, g, h) \propto \Psi_5(f, g, h) \cdot M_{35}(f, g)
\]

The \( \propto \)-symbol indicates that the right-hand side may not add up to one. In that case we just normalize.
Example: Step 3: Message Computation Order

- The structure of the join-tree imposes a partial ordering according to which the messages need to be computed:

\[
M_{41}(b) = \sum_{d} \Psi_{4}(b, d) \\
M_{53}(f, g) = \sum_{h} \Psi_{5}(f, g, h) \\
M_{21}(b, c) = \sum_{a} \Psi_{2}(a, b, c) \\
M_{31}(c, g) = \sum_{f} \Psi_{3}(c, f, g)M_{53}(f, g) \\
M_{13}(c, g) = \sum_{b, e} \Psi_{1}(b, c, e, g)M_{21}(b, c)M_{41}(b) \\
M_{12}(b, c) = \sum_{e, g} \Psi_{2}(b, c, e, g)M_{31}(c, g)M_{41}(b) \\
M_{14}(b) = \sum_{c, e, g} \Psi_{1}(b, c, e, g)M_{21}(b, c)M_{31}(c, g) \\
M_{35}(f, g) = \sum_{c} \Psi_{3}(c, f, g)M_{13}(c, g)
\]

Arrows represent is-needed-for relations. Messages on the same level can be computed in any order. Messages are computed level-wise from top to bottom.
Example: Step 3: Initialization (Potential Layouts)
Example: Step 3: Initialization (Potential Values)

\[ \Psi_1 \]

| \( e_1 \) | \( g_1 \) \( P \) | \( 0.190 \) |
| \( g_2 \) | \( 0.100 \) |
| \( e_2 \) | \( 0.320 \) |
| \( g_2 \) | \( 0.480 \) |

| \( c_1 \) | \( h_1 \) \( P \) | \( 0.2 \) |
| \( h_2 \) | \( 0.8 \) |
| \( e_2 \) | \( 0.350 \) |
| \( g_2 \) | \( 0.350 \) |

| \( c_1 \) | \( h_1 \) \( P \) | \( 0.5 \) |
| \( h_2 \) | \( 0.5 \) |
| \( e_2 \) | \( 0.070 \) |
| \( g_2 \) | \( 0.030 \) |

| \( e_1 \) | \( g_1 \) \( P \) | \( 0.210 \) |
| \( g_2 \) | \( 0.090 \) |
| \( e_2 \) | \( 0.350 \) |
| \( g_2 \) | \( 0.350 \) |

| \( f_1 \) | \( h_1 \) \( P \) | \( 0.4 \) |
| \( h_2 \) | \( 0.6 \) |
| \( g_2 \) | \( 0.7 \) |
| \( f_2 \) | \( h_2 \) | \( 0.3 \) |

| \( \Psi_2 \) | \( P \) |
| \( b_1 \) | \( e_1 \) \( 0.036 \) |
| \( e_2 \) | \( 0.084 \) |
| \( c_1 \) | \( 0.144 \) |
| \( c_2 \) | \( 0.336 \) |

| \( b_2 \) | \( e_1 \) | \( 0.028 \) |
| \( e_2 \) | \( 0.012 \) |
| \( c_1 \) | \( 0.252 \) |
| \( c_2 \) | \( 0.108 \) |

| \( \Psi_3 \) | \( P \) |
| \( c_1 \) | \( g_1 \) | \( 0.1 \) |
| \( g_2 \) | \( 0.1 \) |
| \( f_2 \) | \( g_1 \) | \( 0.9 \) |
| \( g_2 \) | \( 0.9 \) |

| \( \Psi_4 \) | \( P \) |
| \( b_1 \) | \( d_1 \) | \( 0.4 \) |
| \( d_2 \) | \( 0.6 \) |
| \( b_2 \) | \( d_1 \) | \( 0.7 \) |
| \( d_2 \) | \( 0.3 \) |
Example: Step 3: Initialization (Sending Messages)

\[
\begin{align*}
\Psi_2 & \quad P \\
| b_1 & c_1 0.036 \\
| c_2 0.084 \\
| b_2 c_1 0.144 \\
| c_2 0.336 \\
| b_1 c_1 0.028 \\
| c_2 0.012 \\
| b_2 c_1 0.252 \\
| c_2 0.108 \\
\psi_1 & \quad P \\
| c_1 g_1 0.190 \\
| g_2 0.010 \\
| g_1 0.320 \\
| g_2 0.480 \\
| c_2 g_1 0.380 \\
| g_2 0.020 \\
| g_1 0.240 \\
| g_2 0.360 \\
| b_1 g_1 0.210 \\
| g_2 0.090 \\
| c_1 g_1 0.350 \\
| g_2 0.350 \\
| b_2 g_1 0.070 \\
| g_2 0.030 \\
| c_1 g_1 0.450 \\
| g_2 0.450 \\
\end{align*}
\]

\[
\begin{align*}
\Psi_4 & \quad P \\
| b_1 d_1 0.4 \\
| d_2 0.6 \\
| b_2 d_1 0.7 \\
| d_2 0.3 \\
\end{align*}
\]

\[
\begin{align*}
\Psi_3 & \quad P \\
| c_1 g_1 0.1 \\
| g_2 0.1 \\
| f_2 g_1 0.9 \\
| g_2 0.9 \\
| f_1 g_1 0.4 \\
| g_2 0.4 \\
| f_2 g_1 0.6 \\
| g_2 0.6 \\
\end{align*}
\]

\[
\begin{align*}
\Psi_5 & \quad P \\
| f_1 h_1 0.2 \\
| h_2 0.8 \\
| g_1 h_2 0.5 \\
| h_2 0.5 \\
| g_1 h_1 0.4 \\
| h_2 0.6 \\
| g_1 h_1 0.7 \\
| h_2 0.3 \\
\end{align*}
\]

\[
\begin{align*}
M_{21} &= (b_1, c_1, b_2, c_1, b_2, c_2) = (0.06, 0.10, 0.40, 0.44) \\
M_{41} &= (b_1, b_2) = (1, 1)
\end{align*}
\]
Example: Step 3: Initialization (Sending Messages)

\[ M_{21} = (b_1, c_1, b_1, c_2, b_2, c_1, b_2, c_2, 0.06, 0.1, 0.4, 0.44) \]

\[ M_{41} = (b_1, b_2, 1, 1) \]

\[ M_{13} = (c_1, g_1, c_1, g_2, c_2, g_1, c_2, g_2, 0.254, 0.206, 0.29, 0.25) \]

\[ M_{35} = (f_1, g_1, f_1, g_2, f_2, g_1, f_2, g_2, 0.14, 0.12, 0.4, 0.33) \]
Example: Step 3: Initialization (Sending Messages)

\[
\begin{align*}
\Psi_2 & \quad P \\
| a_1 & c_1 & 0.036 & e_1 \\
& c_2 & 0.084 & e_2 \\
| b_1 & c_1 & 0.144 \\
& c_2 & 0.336 \\
| b_2 & c_1 & 0.028 & e_1 \\
& c_2 & 0.012 & e_2 \\
| b_2 & c_1 & 0.252 \\
& c_2 & 0.108 \\
\end{align*}
\]

\[
\begin{align*}
\Psi_4 & \quad P \\
| b_1 & d_1 & 0.4 \\
& d_2 & 0.6 \\
| b_2 & d_1 & 0.7 \\
& d_2 & 0.3 \\
\end{align*}
\]

\[
\begin{align*}
M_{21} &= (b_1, c_1, b_1, c_2, b_2, c_1, b_2, c_2) = (0.06, 0.10, 0.40, 0.44) \\
M_{41} &= (b_1, b_2) = (1, 1) \\
M_{13} &= (c_1, g_1, c_1, g_2, c_2, g_1, c_2, g_2) = (0.254, 0.206, 0.290, 0.250) \\
M_{35} &= (f_1, g_1, f_1, g_2, f_2, g_1, f_2, g_2) = (0.14, 0.12, 0.40, 0.33) \\
M_{53} &= (f_1, g_1, f_1, g_2, f_2, g_1, f_2, g_2) = (1, 1, 1, 1) \\
M_{31} &= (c_1, g_1, c_1, g_2, c_2, g_1, c_2, g_2) = (1, 1, 1, 1)
\end{align*}
\]
Example: Step 3: Initialization (Sending Messages)

\[ M_{21} = (b_1c_1, b_1c_2, b_2c_1, b_2c_2, 0.06, 0.10, 0.40, 0.44) \]

\[ M_{41} = (b_1, b_2) \]

\[ M_{13} = (c_1g_1, c_1g_2, c_2g_1, c_2g_2, 0.254, 0.206, 0.290, 0.250) \]

\[ M_{35} = (f_1g_1, f_1g_2, f_2g_1, f_2g_2, 0.14, 0.12, 0.40, 0.33) \]

\[ M_{53} = (f_1g_1, f_1g_2, f_2g_1, f_2g_2, 0.1, 0.1, 0.1, 0.1) \]

\[ M_{31} = (c_1g_1, c_1g_2, c_2g_1, c_2g_2, 0.1, 0.1, 0.1, 0.1) \]

\[ M_{12} = (b_1c_1, b_1c_2, b_2c_1, b_2c_2, 0.16, 0.84) \]
**Example: Step 3: Initialization Complete**

$$\Psi_2$$

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<tr>
<th>$a_1$</th>
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$$\Psi_1$$

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<td>$g_2$</td>
<td>0.010</td>
<td>0.0006</td>
</tr>
<tr>
<td></td>
<td>$g_1$</td>
<td>0.320</td>
<td>0.0205</td>
</tr>
<tr>
<td></td>
<td>$g_2$</td>
<td>0.480</td>
<td>0.0307</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$e_2$</th>
<th>$g_1$</th>
<th>0.380</th>
<th>0.0365</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$g_2$</td>
<td>0.200</td>
<td>0.0019</td>
</tr>
<tr>
<td></td>
<td>$g_1$</td>
<td>0.240</td>
<td>0.0230</td>
</tr>
<tr>
<td></td>
<td>$g_2$</td>
<td>0.360</td>
<td>0.0346</td>
</tr>
</tbody>
</table>

**Matrixes:**

$$M_{21} = (b_1, c_1, b_2, c_2, b_2, c_2)$$

$$M_{41} = (b_1, b_2)$$

$$M_{13} = (c_1, c_2, g_1, c_2, g_2, g_2)$$

$$M_{35} = (f_1, g_1, f_1, g_2, f_2, g_1, f_2, g_2)$$

$$M_{53} = (f_1, g_1, f_1, g_2, f_2, g_1, f_2, g_2)$$

$$M_{31} = (c_1, c_2, g_1, c_2, g_2, g_2)$$

$$M_{12} = (b_1, c_1, b_1, c_2, b_2, c_1, b_2, c_2)$$

$$M_{14} = (b_1, b_2)$$

$$P \times A = B \times C \times D \times E \times F \times G \times H$$

<table>
<thead>
<tr>
<th>$P$</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
<th>$E$</th>
<th>$F$</th>
<th>$G$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.6000</td>
<td>0.1600</td>
<td>0.4600</td>
<td>0.6520</td>
<td>0.2144</td>
<td>0.2620</td>
<td>0.5448</td>
<td>0.4842</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4000</td>
<td>0.8400</td>
<td>0.4500</td>
<td>0.3480</td>
<td>0.7856</td>
<td>0.7380</td>
<td>0.4552</td>
<td>0.5158</td>
</tr>
</tbody>
</table>
Example: Step 4: Evidence $H = h_1$ (Altering Potentials)
Example: Step 4: Evidence $H = h_1$ (Sending Messages)

$$M_{53} = (f_{1,g_1} f_{1,g_2} f_{2,g_1} f_{2,g_2})$$
Example: Step 4: Evidence $H = h_1$ (Sending Messages)

\[
M_{53} = \left( f_1, g_1, f_1, g_2, f_2, g_1, f_2, g_2 \right)
\]

\[
M_{21} = \left( b_1, c_1, b_1, c_2, b_2, c_1, b_2, c_2 \right)
\]

\[
M_{41} = \left( b_1, b_2 \right)
\]
Example: Step 4: Evidence $H = h_1$ (Sending Messages)

\[
M_{53} = \begin{pmatrix}
  f_1, g_1 & f_1, g_2 & f_2, g_1 & f_2, g_2
\end{pmatrix} = \begin{pmatrix}
  0.2 & 0.5 & 0.4 & 0.7
\end{pmatrix}
\]

\[
M_{21} = \begin{pmatrix}
  b_1, c_1 & b_1, c_2 & b_2, c_1 & b_2, c_2
\end{pmatrix} = \begin{pmatrix}
  0.06 & 0.10 & 0.40 & 0.44
\end{pmatrix}
\]

\[
M_{41} = \begin{pmatrix}
  b_1 & b_2
\end{pmatrix} = \begin{pmatrix}
  1 & 1
\end{pmatrix}
\]

\[
M_{31} = \begin{pmatrix}
  c_1, g_1 & c_1, g_2 & c_2, g_1 & c_2, g_2
\end{pmatrix} = \begin{pmatrix}
  0.38 & 0.68 & 0.32 & 0.62
\end{pmatrix}
\]
Example: Step 4: Evidence $H = h_1$ (Sending Messages)

\[
\begin{array}{c|c|c}
\Psi_2 & a_1 & b_1 \\
\hline
 & c_1 & 0.036 \\
 & c_2 & 0.084 \\
\hline
 & c_1 & 0.144 \\
 & c_2 & 0.336 \\
\hline
 & c_1 & 0.028 \\
 & c_2 & 0.012 \\
\hline
 & c_1 & 0.252 \\
 & c_2 & 0.108 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\Psi_1 & \Psi_2 & \Psi_3 \\
\hline
 & a_1 & b_1 \\
\hline
 & c_1 & 0.190 \\
 & g_1 & 0.010 \\
 & g_2 & 0.320 \\
\hline
 & c_2 & 0.480 \\
 & g_1 & 0.380 \\
 & g_2 & 0.420 \\
\hline
 & c_1 & 0.240 \\
 & g_1 & 0.280 \\
 & g_2 & 0.360 \\
\hline
 & c_2 & 0.210 \\
 & g_1 & 0.090 \\
 & g_2 & 0.090 \\
\hline
 & c_1 & 0.350 \\
 & g_1 & 0.070 \\
 & g_2 & 0.350 \\
\hline
 & c_2 & 0.450 \\
 & g_1 & 0.450 \\
 & g_2 & 0.450 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\Psi_4 & \Psi_3 & \Psi_5 \\
\hline
 & b_1 & d_1 \\
 & d_2 & 0.6 \\
\hline
 & b_2 & d_1 & 0.7 \\
 & d_2 & 0.3 \\
\hline
 & f_1 & g_1 & 0.1 \\
 & g_2 & 0.1 \\
\hline
 & f_2 & g_1 & 0.9 \\
 & g_2 & 0.9 \\
\hline
 & f_1 & g_1 & 0.4 \\
 & g_2 & 0.4 \\
\hline
 & f_2 & g_1 & 0.6 \\
 & g_2 & 0.6 \\
\hline
 & g_1 & h_1 & 0.2 \\
 & h_2 & 0 \\
\hline
 & g_2 & h_1 & 0.5 \\
 & h_2 & 0 \\
\hline
 & g_1 & h_1 & 0.4 \\
 & h_2 & 0 \\
\hline
 & g_2 & h_1 & 0.7 \\
 & h_2 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
M_53 & f_1.g_1 & f_1.g_2 \\
& f_2.g_1 & f_2.g_2 \\
\hline
M_21 & b_1.c_1 & b_1.c_2 \\
& b_2.c_1 & b_2.c_2 \\
\hline
M_{41} & b_1 & b_2 \\
\hline
M_{31} & c_1.g_1 & c_1.g_2 \\
& c_2.g_1 & c_2.g_2 \\
\hline
M_{12} & b_1.c_1 & b_1.c_2 \\
& b_2.c_1 & b_2.c_2 \\
\hline
M_{14} & b_1 & b_2 \\
\hline
M_{13} & c_1.g_1 & c_1.g_2 \\
& c_2.g_1 & c_2.g_2 \\
\hline
M_{35} & f_1.g_1 & f_1.g_2 \\
& f_2.g_1 & f_2.g_2 \\
\end{array}
\]
Example: Step 4: Evidence $H = h_1$ Incorporated

\[M_{53} = \begin{pmatrix} 0.2 & 0.5 & 0.4 & 0.7 \end{pmatrix}\]
\[M_{21} = \begin{pmatrix} 0.06 & 0.10 & 0.40 & 0.44 \end{pmatrix}\]
\[M_{41} = \begin{pmatrix} b_1 & b_2 \end{pmatrix}\]
\[M_{31} = \begin{pmatrix} 0.38 & 0.68 & 0.32 & 0.62 \end{pmatrix}\]
\[M_{12} = \begin{pmatrix} 0.527 & 0.434 & 0.512 & 0.464 \end{pmatrix}\]
\[M_{14} = \begin{pmatrix} b_1 & b_2 \end{pmatrix}\]
\[M_{13} = \begin{pmatrix} 0.254 & 0.206 & 0.290 & 0.250 \end{pmatrix}\]
\[M_{35} = \begin{pmatrix} 0.14 & 0.12 & 0.40 & 0.33 \end{pmatrix}\]
Manual creation of a reasoning system based on a graphical model:

- causal model of given domain
- conditional independence graph
- decomposition of the distribution
- evidence propagation scheme

- Problem: strong assumptions about the statistical effects of causal relations.
- Nevertheless this approach often yields usable graphical models.
Example 1: Genotype Determination of Danish Jersey Cattle

Assumptions about parents:
- risk about misstatement

Genotype mother (dam)

Genotype father (sire)

Genotype child:
- 6 possible values

4 lysis values measured by photometer

Reliability of databases

Inheritance rules

Blood group determination
Example 1: Genotype Determination of Danish Jersey Cattle

Danish Jersey Cattle Blood Type Determination

The grey nodes correspond to observable attributes.

- This graph was specified by human domain experts, based on knowledge about (causal) dependences of the variables.
Example 1: Genotype Determination of Danish Jersey Cattle

- Full 21-dimensional domain has $2^6 \cdot 3^{10} \cdot 6 \cdot 8^4 = 92,876,046,336$ possible states.
- Bayesian network requires only 306 conditional probabilities.
- Example of a conditional probability table (attributes 2, 9, and 5):

<table>
<thead>
<tr>
<th>sire correct</th>
<th>true sire phenogroup 1</th>
<th>stated sire phenogroup 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>yes</td>
<td>F1</td>
<td>1 0 0</td>
</tr>
<tr>
<td>yes</td>
<td>V1</td>
<td>0 1 0</td>
</tr>
<tr>
<td>yes</td>
<td>V2</td>
<td>0 0 1</td>
</tr>
<tr>
<td>no</td>
<td>F1</td>
<td>0.58 0.10 0.32</td>
</tr>
<tr>
<td>no</td>
<td>V1</td>
<td>0.58 0.10 0.32</td>
</tr>
<tr>
<td>no</td>
<td>V2</td>
<td>0.58 0.10 0.32</td>
</tr>
</tbody>
</table>

- The probabilities are acquired from human domain experts or estimated from historical data.
Example 1: Genotype Determination of Danish Jersey Cattle

moral graph
(already triangulated)

join tree
Marginal distributions before setting evidence:
Conditional distributions given evidence in the input variables:
Example 2: Item Planning at Volkswagen

Strategy of the VW Group

<table>
<thead>
<tr>
<th>Marketing strategy</th>
<th>Vehicle specification by clients</th>
<th>Bestsellers defined by manufacturer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complexity</td>
<td>Huge number of variants</td>
<td>Small number of variants</td>
</tr>
</tbody>
</table>

Vehicle specification

<table>
<thead>
<tr>
<th>Equipment</th>
<th>fastback</th>
<th>2.81, 150 kW</th>
<th>Type Alpha</th>
<th>4</th>
<th>leather</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group</td>
<td>car body type</td>
<td>engine</td>
<td>radio</td>
<td>doors</td>
<td>seat cover</td>
<td>...</td>
</tr>
</tbody>
</table>
Example 2: Model “Golf”

- Approx. 200 equipment groups
- 2 to 50 items per group
- Therefore more than $2^{200}$ possible vehicle specifications
- Choice of valid specifications is constrained by a rule system (10000 technical rules, plus marketing and production rules)

Example of technical rules:

- **If** Engine=$e_1$ **then** Transmission=$t_3$

- **If** Engine=$e_4$ and Heating=$h_2$ **then** Generator $\in \{g_3, g_4, g_5\}$
Problem Representation

Historical Data
Sample of produced *vehicle specifications*
(representative choice, context-dependent, e.g. Golf)

System of Rules
*Rules* for the validity of item combinations
(specified for a vehicle class and a planning interval)

Prediction & Planning
Predicted / assigned *planning data*
.production program, demands, installation rates, capacity restrictions, ...
Complexity of the Planning Problem

Equipment table

<table>
<thead>
<tr>
<th></th>
<th>Engine</th>
<th>Transmission</th>
<th>Heating</th>
<th>Generator</th>
<th>…</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$e_1$</td>
<td>$t_3$</td>
<td>$h_1$</td>
<td>$g_1$</td>
<td>…</td>
</tr>
<tr>
<td>2</td>
<td>$e_2$</td>
<td>$t_4$</td>
<td>$h_3$</td>
<td>$g_5$</td>
<td>…</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
</tr>
<tr>
<td>100000</td>
<td>$e_7$</td>
<td>$t_1$</td>
<td>$h_3$</td>
<td>$g_2$</td>
<td>…</td>
</tr>
</tbody>
</table>

Installation rates

<table>
<thead>
<tr>
<th></th>
<th>Engine</th>
<th>Transmission</th>
<th>Heating</th>
<th>Generator</th>
<th>…</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>$t_1$</td>
<td>$h_1$</td>
<td>$g_1$</td>
<td>…</td>
<td></td>
<td>0.0000012</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
<td></td>
<td>…</td>
</tr>
</tbody>
</table>

Result is a 200-dimensional, finite probability space

- $P(\text{Engine} = e_1, \text{Transmission} = t_3) = ?$
- $P(\text{Heating} = h_1 \mid \text{Generator} = g_3) = ?$  
  Problem of complexity!
Solution: Decomposition into Subspaces

\[ P(E, H, T, A) = P(A | E, H, T) \cdot P(T | E, H) \cdot P(E | H) \cdot P(H) \]

\[ \text{here } P(A | E, H) \cdot P(T | E) \cdot P(E) \cdot P(H) \]

Bayesian Network

\[ P(E, H, T, A) = P(A | E, H, T) \cdot P(T | E, H) \cdot P(E | H) \cdot P(H) \]

\[ \text{here } P(A | E, H) \cdot P(T | E) \cdot P(E) \cdot P(H) \]

Hypergraph Decomposition
Clique Tree of the VW Bora
Typical Planning Operation: Focusing

- **Application:**
  - Compute item demand
    Calculation of installation rates of equipment combinations
  - Simulation
    Analyze customer requirements (e.g. of persons having ordered a navigation system for a VW Polo)

- **Input:** Equipment combinations

- **Operation:** Compute
  - the conditional network distribution and
  - the probabilities of the specified equipment combinations.
Implementation and Deployment

- Projec leader: Jörg Gebhardt
- Client server system
- Server on 6–8 machines
- Quadcore platform
- Terabyte hard drive
- Java, Linux, Oracle
- WebSphere application server
- Software used daily worldwide
- 15 developers
- 4000 Bayesian networks are currently used