OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF FAKULTÄT FÜR INFORMATIK

Fuzzy Systems

Fuzzy Logic

Prof. Dr. Rudolf Kruse   Christoph Doell

{kruse,doell}@iws.cs.uni-magdeburg.de
Otto-von-Guericke University of Magdeburg
Faculty of Computer Science
Department of Knowledge Processing and Language Engineering

Outline

1. Complement

Strict and Strong Negations

Families of Negations

Representation of Negations

2. Intersection and Union

3. Implication

Fuzzy Complement/Fuzzy Negation

Definition

Let X be a given set and $\mu \in \mathcal{F}(X)$. Then the *complement* $\bar{\mu}$ can be defined pointwise by $\bar{\mu}(x) :=\sim (\mu(x))$ where $\sim : [0,1] \to [0,1]$ satisfies the conditions

$$\sim (0) = 1, \quad \sim (1) = 0$$

and

$$\text{for } x, y \in [0,1],\ x \leq y \implies \sim x \geq \sim y \quad (\sim \text{ is non-increasing}).$$

Abbreviation: $\sim x :=\sim (x)$

Strict and Strong Negations

Additional properties may be required

- $x, y \in [0, 1]$, $x < y \Longrightarrow \sim x > \sim y$ ($\sim$ is strictly decreasing)
- $\sim$ is continuous
- $\sim \sim x = x$ for all $x \in [0, 1]$ ($\sim$ is involutive)

According to conditions, two subclasses of negations are defined:

Definition

A negation is called *strict* if it is also strictly decreasing and continuous. A strict negation is said to be *strong* if it is involutive, too.

$\sim x = 1 - x^2$, for instance, is strict, not strong, thus not involutive

Families of Negations

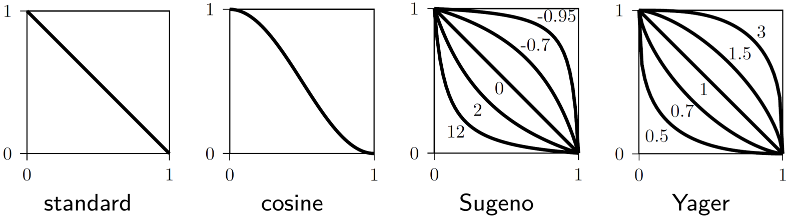
standard negation: $\sim x = 1 - x$

threshold negation: $\sim_\theta (x) = \begin{cases} 1 & \text{if } x \le \theta \\ 0 & \text{otherwise} \end{cases}$

Cosine negation: $\sim x = \frac{1}{2}(1 + \cos(\pi x))$

Sugeno negation: $\sim_\lambda (x) = \frac{1 - x}{1 + \lambda x}, \quad \lambda > -1$

Yager negation: $\sim_\lambda (x) = (1 - x^\lambda)^{\frac{1}{\lambda}}$



Two Extreme Negations

$$\textit{intuitionistic negation} \quad \sim_i (x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

$$\textit{dual intuitionistic negation} \quad \sim_{di} (x) = \begin{cases} 1 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

Both negations are not strictly increasing, not continuous, not involutive

Thus they are neither strict nor strong

They are "optimal" since their notions are nearest to crisp negation

$\sim_i$ and $\sim_{di}$ are two extreme cases of negations

For any negation $\sim$ the following holds

$$\sim_i \leq \sim \leq \sim_{di}$$

Inverse of a Strict Negation

Any strict negation $\sim$ is strictly decreasing and continuous.

Hence one can define its inverse $\sim^{-1}$.

$\sim^{-1}$ is also strict but in general differs from $\sim$.

$\sim^{-1}=\sim$ if and only if $\sim$ is involutive.

Every strict negation $\sim$ has a unique value $0 < s_\sim < 1$ such that $\sim s_\sim = s_\sim$.

$s_\sim$ is called *membership crossover point*.

$A(a) > s_\sim$ if and only if $A^c(a) < s_\sim$ where A^c is defined via $\sim$.

$\sim^{-1}(s_\sim) = s_\sim$ always holds as well.

Representation of Negations

Any strong negation can be obtained from standard negation.

Let $a, b \in \mathbb{R},\ a \leq b$.

Let $\varphi : [a, b] \to [a, b]$ be continuous and strictly increasing.

φ is called *automorphism* of the interval $[a, b] \subset \mathbb{R}$.

Theorem

A function $\sim : [0, 1] \to [0, 1]$ *is a strong negation if and only if there exists an automorphism* φ *of the unit interval such that for all* $x \in [0, 1]$ *the following holds*

$$\sim_\varphi (x) = \varphi^{-1}(1 - \varphi(x)).$$

$\sim_\varphi (x) = \varphi^{-1}(1 - \varphi(x))$ is called φ-transform of the standard negation.

Outline

Classical Intersection and Union

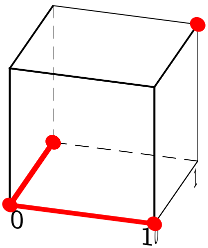
Classical set intersection represents logical conjunction.

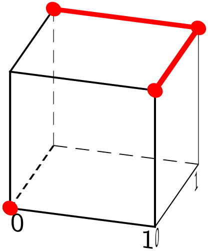
Classical set union represents logical disjunction.

Generalization from $\{0, 1\}$ to $[0, 1]$ as follows:

$x \wedge y$	0	1
0	0	0
1	0	1

$x \vee y$	0	1
0	0	1
1	1	1





Fuzzy Intersection and Union

Let A, B be fuzzy subsets of X, *i.e.* $A, B \in \mathcal{F}(X)$.

Their **intersection** and **union** can be defined pointwise using:

$$(A \cap B)(x) = \top(A(x), B(x)) \quad \text{where} \quad \top : [0,1]^2 \to [0,1]$$
$$(A \cup B)(x) = \bot(A(x), B(x)) \quad \text{where} \quad \bot : [0,1]^2 \to [0,1].$$

Triangular Norms and Conorms I

$\top$ is a *triangular norm (t-norm)* $\Longleftrightarrow$ $\top$ satisfies conditions T1-T4

$\bot$ is a *triangular conorm (t-conorm)* $\Longleftrightarrow$ $\bot$ satisfies C1-C4

for all $x, y \in [0, 1]$, the following laws hold

Identity Law

T1: $\top(x, 1) = x \quad (A \cap X = A)$
C1: $\bot(x, 0) = x \quad (A \cup \emptyset = A)$.

Commutativity

T2: $\top(x, y) = \top(y, x) \quad (A \cap B = B \cap A)$,
C2: $\bot(x, y) = \bot(y, x) \quad (A \cup B = B \cup A)$.

Triangular Norms and Conorms II

for all $x, y, z \in [0, 1]$, the following laws hold

Associativity

T3: $\top(x, \top(y, z)) = \top(\top(x, y), z) \quad (A \cap (B \cap C) = (A \cap B) \cap C)$,
C3: $\bot(x, \bot(y, z)) = \bot(\bot(x, y), z) \quad (A \cup (B \cup C) = (A \cup B) \cup C)$.

Monotonicity

$y \leq z$ implies
T4: $\top(x, y) \leq \top(x, z)$
C4: $\bot(x, y) \leq \bot(x, z)$.

Triangular Norms and Conorms III

$\top$ is a *triangular norm (t-norm)* $\iff$ $\top$ satisfies conditions T1-T4

$\bot$ is a *triangular conorm (t-conorm)* $\iff$ $\bot$ satisfies C1-C4

Both identity law and monotonicity respectively imply

$$\forall x \in [0,1] : \top(0,x) = 0,$$
$$\forall x \in [0,1] : \bot(1,x) = 1,$$
$$\text{for any } t\text{-norm } \top : \top(x,y) \leq \min(x,y),$$
$$\text{for any } t\text{-conorm } \bot : \bot(x,y) \geq \max(x,y).$$

note: $x = 1 \Rightarrow T(0,1) = 0$ and
$x \leq 1 \Rightarrow T(x,0) \leq T(1,0) = T(0,1) = 0$

De Morgan Triplet I

For every $\top$ and strong negation $\sim$, one can define t-conorm $\bot$ by

$$\bot(x, y) = \sim \top(\sim x, \sim y), \quad x, y \in [0, 1].$$

Additionally, in this case $\top(x, y) = \sim \bot(\sim x, \sim y), \ \ x, y \in [0, 1]$.

$\bot, \top$ are called *N-dual t-conorm* and *N-dual t-norm* to $\top, \bot$, resp.

In case of the standard negation $\sim x = 1 - x$ for $x \in [0, 1]$,
N-dual $\bot$ and $\top$ are called *dual t-conorm* and *dual t-norm*, resp.

$\bot(x, y) = \sim \top(\sim x, \sim y)$ expresses "fuzzy " De Morgan's law.

note: De Morgan's laws $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

De Morgan Triplet II

Definition

The triplet $(\top, \bot, \sim)$ is called *De Morgan triplet* if and only if

$\top$ is t-norm, $\bot$ is t-conorm, $\sim$ is strong negation,

$\top, \bot$ and $\sim$ satisfy $\bot(x, y) = \sim \top(\sim x, \sim y)$.

In the following, some important De Morgan triplets will be shown,

only the most frequently used and important ones.

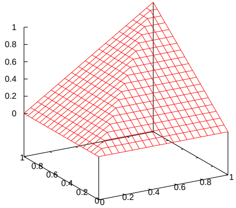
In all cases, the standard negation $\sim x = 1 - x$ is considered.

The Minimum and Maximum I

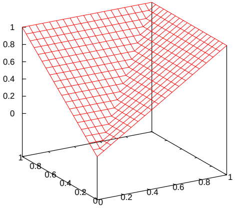
$\top_{\min}(x, y) = \min(x, y), \quad \bot_{\max}(x, y) = \max(x, y)$

Minimum is the greatest *t*-norm and max is the weakest *t*-conorm.

$\top(x, y) \leq \min(x, y)$ and $\bot(x, y) \geq \max(x, y)$ for any $\top$ and $\bot$



$\top_{\min}$



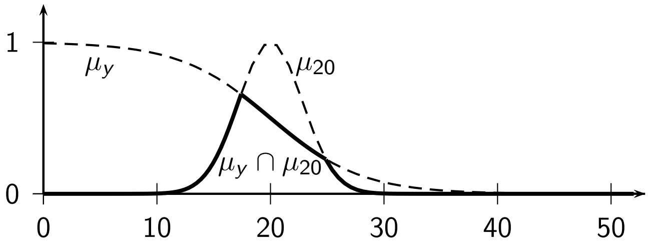
$\bot_{\max}$

The Minimum and Maximum II

$\top_{\text{min}}$ and $\bot_{\text{max}}$ can be easily processed numerically and visually,

e.g. linguistic values *young* and *approx. 20* described by μ_y, μ_{20}.

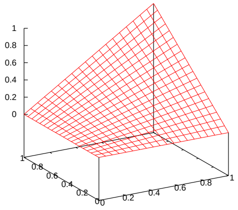
$\top_{\text{min}}(\mu_y, \mu_{20})$ is shown below.



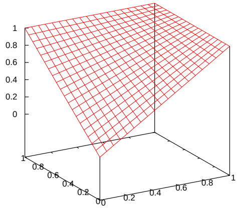
The Product and Probabilistic Sum

$$\top_{\text{prod}}(x, y) = x \cdot y, \quad \bot_{\text{sum}}(x, y) = x + y - x \cdot y$$

Note that use of product and its dual has nothing to do with probability theory.



$\top_{\text{prod}}$

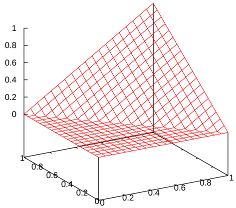


$\bot_{\text{sum}}$

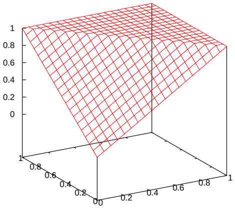
The Łukasiewicz t-norm and t-conorm

$$\top_{\text{Łuka}}(x, y) = \max\{0,\ x + y - 1\}, \quad \bot_{\text{Łuka}}(x, y) = \min\{1,\ x + y\}$$

$\top_{\text{Łuka}}$, $\bot_{\text{Łuka}}$ are also called *bold intersection* and *bounded sum*.



$\top_{\text{Łuka}}$



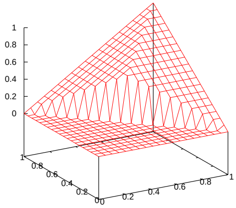
$\bot_{\text{Łuka}}$

The Nilpotent Minimum and Maximum

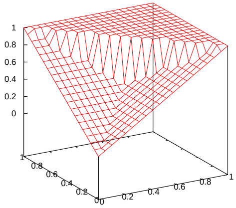
$$\top_{\min_0}(x, y) = \begin{cases} \min(x, y) & \text{if } x + y > 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\bot_{\max_1}(x, y) = \begin{cases} \max(x, y) & \text{if } x + y < 1 \\ 1 & \text{otherwise} \end{cases}$$

New since found in 1992 and independently rediscovered in 1995.



$\top_{\min_0}$



$\bot_{\max_1}$

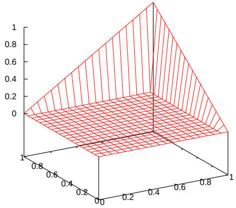
The Drastic Product and Sum

$$\top_{-1}(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1 \\ 0 & \text{otherwise} \end{cases}$$

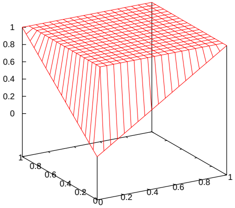
$$\bot_{-1}(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y) = 0 \\ 1 & \text{otherwise} \end{cases}$$

$\top_{-1}$ is the weakest t-norm, $\bot_{-1}$ is the strongest t-conorm.

$\top_{-1} \leq \top \leq \top_{\min}, \quad \bot_{\max} \leq \bot \leq \bot_{-1}$ for any $\top$ and $\bot$

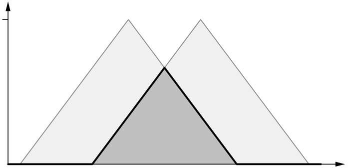


$\top_{-1}$

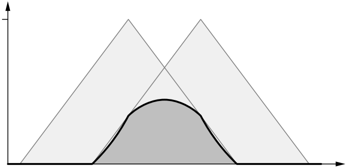


$\bot_{-1}$

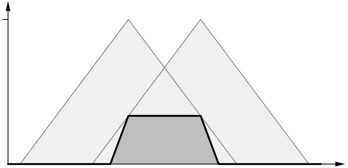
Examples of Fuzzy Intersections



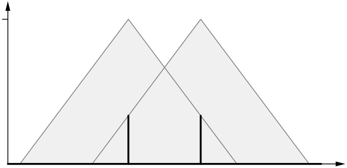
t-norm $\top_{\min}$



t-norm $\top_{\text{prod}}$



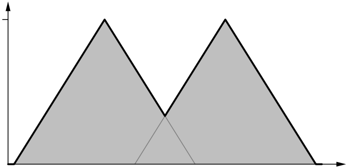
t-norm $\top_{\text{Łuka}}$



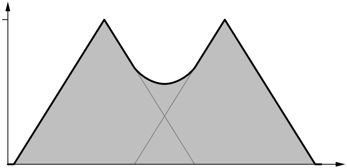
t-norm $\top_{-1}$

Note that all fuzzy intersections are contained within upper left graph and lower right one.

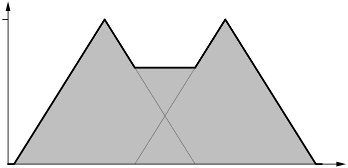
Examples of Fuzzy Unions



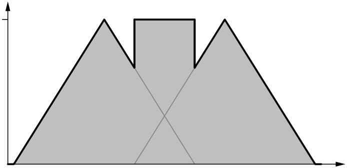
t-conorm $\bot_{\max}$



t-conorm $\bot_{\text{sum}}$



t-conorm $\bot_{\text{Łuka}}$



t-conorm $\bot_{-1}$

Note that all fuzzy unions are contained within upper left graph and lower right one.

The Special Role of Minimum and Maximum I

$\top_{\min}$ and $\bot_{\max}$ play key role for intersection and union, resp.

In a practical sense, they are very simple.

Apart from the identity law, commutativity, associativity and monotonicity, they also satisfy the following properties for all $x, y, z \in [0, 1]$:

Distributivity

$$\bot_{\max}(x, \top_{\min}(y, z)) = \top_{\min}(\bot_{\max}(x, y), \bot_{\max}(x, z)),$$
$$\top_{\min}(x, \bot_{\max}(y, z)) = \bot_{\max}(\top_{\min}(x, y), \top_{\min}(x, z))$$

Continuity

$\top_{\min}$ and $\bot_{\max}$ are continuous.

The Special Role of Minimum and Maximum II

Strict monotonicity on the diagonal

$x < y$ implies $\top_{\min}(x, x) < \top_{\min}(y, y)$ and $\bot_{\max}(x, x) < \bot_{\max}(y, y)$.

Idempotency

$$\top_{\min}(x, x) = x, \quad \bot_{\max}(x, x) = x$$

Absorption

$$\top_{\min}(x, \bot_{\max}(x, y)) = x, \quad \bot_{\max}(x, \top_{\min}(x, y)) = x$$

Non-compensation

$x < y < y$ imply $\top_{\min}(x, z) \neq \top_{\min}(y, y)$ and $\bot_{\max}(x, z) \neq \bot_{\max}(y, y)$.

The Special Role of Minimum and Maximum III

Is $(\mathcal{F}(X), \top_{\min}, \bot_{\max}, \sim)$ a boolean algebra?

Consider the properties (B1)-(B9) of any Boolean algebra.

For $(\mathcal{F}(X), \top_{\min}, \bot_{\max}, \sim)$ with strong negation $\sim$ only complementary (B7) does not hold.

Hence $(\mathcal{F}(X), \top_{\min}, \bot_{\max}, \sim)$ is a *completely distributive lattice* with identity element μ_X and zero element $\mu_\emptyset$.

No lattice $(\mathcal{F}(X), \top, \bot, \sim)$ forms a Boolean algebra

due to the fact that complementary (B7) does not hold:

- There is no complement/negation $\sim$ with $\top(A, \sim A) = \mu_\emptyset$.
- There is no complement/negation $\sim$ with $\bot(A, \sim A) = \mu_X$.

Complementary Property of Fuzzy Sets I

Using fuzzy sets, it's **impossible** to keep up a Boolean algebra.

Verify, *e.g.* that law of contradiction is violated, *i.e.*

$$(\exists x \in X)(A \cap A^c)(x) \neq \emptyset.$$

We use $\min, \max$ and strong negation $\sim$ as fuzzy set operators.

So we need to show that

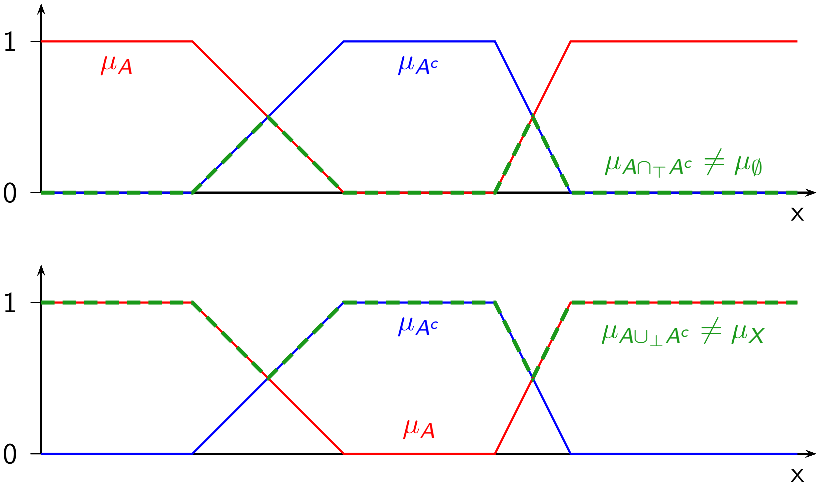
$$\min\{A(x), 1 - A(x)\} = 0$$

is violated for at least one $x \in X$.

easy: This Equation is violated for all $A(x) \in (0, 1)$.

It is satisfied only for $A(x) \in \{0, 1\}$.

Complementary Property of Fuzzy Sets II: Example



The concept of a pseudoinverse

Definition

Let $f : [a, b] \to [c, d]$ be a monotone function between two closed subintervals of extended real line. The pseudoinverse function to f is the function $f^{(-1)} : [c, d] \to [a, b]$ defined as

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [a, b] \mid f(x) < y\} & \text{for } f \text{ non-decreasing,} \\ \sup\{x \in [a, b] \mid f(x) > y\} & \text{for } f \text{ non-increasing.} \end{cases}$$

Continuous Archimedean t-norms and t-conorms

broad class of problems relates to representation of multi-place functions by composition of "simpler" functions, *e.g.*

$$K(x, y) = g(f(x) + f(y))$$

So, one should consider suitable subclass of all t-norms.

Definition

A t-norm $\top$ is

(a) *continuous* if $\top$ as function is continuous on unit interval,

(b) *Archimedean* if $\top$ is continuous and $\top(x, x) < x$ for all $x \in]0, 1[$.

Definition

A t-conorm $\bot$ is

(a) *continuous* if $\bot$ as function is continuous on unit interval,

(b) *Archimedean* if $\bot$ is continuous and $\bot(x, x) > x$ for all $x \in]0, 1[$.

Continuous Archimedean t-norms

Theorem

A t-norm $\top$ is continuous and Archimedean if and only if there exists a strictly decreasing and continuous function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$ such that

$$\top(x,y) = f^{(-1)}(f(x) + f(y)) \qquad (1)$$

where

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \text{if } x \leq f(0) \\ 0 & \text{otherwise} \end{cases}$$

is the pseudoinverse of f. Moreover, this representation is unique up to a positive multiplicative constant.

$\top$ *is generated by f* if $\top$ has representation (1).

f is called *additive generator* of $\top$.

Additive Generators of t-norms – Examples

Find an additive generator f of $\top_{\text{Łuka}} = \max\{x + y - 1,\ 0\}$.

for instance $f_{\text{Łuka}}(x) = 1 - x$

then, $f_{\text{Łuka}}^{(-1)}(x) = \max\{1 - x,\ 0\}$

thus $\top_{\text{Łuka}}(x, y) = f_{\text{Łuka}}^{(-1)}(f_{\text{Łuka}}(x) + f_{\text{Łuka}}(y))$

Find an additive generator f of $\top_{\text{prod}} = x \cdot y$.

to be discussed in the exercise

hint: use of logarithmic and exponential function

Continuous Archimedean t-conorms

Theorem

A t-conorm $\perp$ is continuous and Archimedean if and only if there exists a strictly increasing and continuous function $g : [0,1] \to [0,\infty]$ with $g(0) = 0$ such that

$$\perp(x,y) = g^{(-1)}(g(x) + g(y)) \qquad (2)$$

where

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } x \leq g(1) \\ 1 & \text{otherwise} \end{cases}$$

is the pseudoinverse of g. Moreover, this representation is unique up to a positive multiplicative constant.

$\perp$ *is generated by g* if $\perp$ has representation (2).

g is called *additive generator* of $\perp$.

Additive Generators of t-conorms – Two Examples

Find an additive generator g of $\bot_{\textbf{Łuka}} = \min\{x + y,\ 1\}$.

for instance $g_{\text{Łuka}}(x) = x$

then, $g^{(-1)}_{\text{Łuka}}(x) = \min\{x,\ 1\}$

thus $\bot_{\text{Łuka}}(x, y) = g^{(-1)}_{\text{Łuka}}(g_{\text{Łuka}}(x) + g_{\text{Łuka}}(y))$

Find an additive generator g of $\bot_{\textbf{sum}} = x + y - x \cdot y$.

to be discussed in the exercise

hint: use of logarithmic and exponential function

Now, let us examine some typical families of operations.

Hamacher Family I

$$\top_\alpha(x, y) = \frac{x \cdot y}{\alpha + (1-\alpha)(x + y + x \cdot y)}, \quad \alpha \geq 0,$$

$$\bot_\beta(x, y) = \frac{x + y + (\beta - 1) \cdot x \cdot y}{1 + \beta \cdot x \cdot y}, \quad \beta \geq -1,$$

$$\sim_\gamma (x) = \frac{1 - x}{1 + \gamma x}, \quad \gamma > -1$$

Theorem

$(\top, \bot, \sim)$ *is a De Morgan triplet such that*

$$\top(x, y) = \top(x, z) \Longrightarrow y = z,$$

$$\bot(x, y) = \bot(x, z) \Longrightarrow y = z,$$

$$\forall z \leq x \; \exists y, y' \text{ such that } \top(x, y) = z, \; \bot(z, y') = x$$

and $\top$ *and* $\bot$ *are rational functions if and only if there are numbers* $\alpha \geq 0, \; \beta \geq -1$ *and* $\gamma > -1$ *such that* $\alpha = \frac{1+\beta}{1+\gamma}$ *and* $\top = \top_\alpha, \; \bot = \bot_\beta$ *and* $\sim = \sim_\gamma$.

Hamacher Family II

Additive generators f_α of $\top_\alpha$ are

$$f_\alpha = \begin{cases} \frac{1-x}{x} & \text{if } \alpha = 0 \\ \log \frac{\alpha+(1-\alpha)x}{x} & \text{if } \alpha > 0. \end{cases}$$

Each member of these families is strict t-norm and strict t-conorm, respectively.

Members of this family of t-norms are decreasing functions of parameter α.

Sugeno-Weber Family I

For $\lambda > 1$ and $x, y \in [0, 1]$, define

$$\top_\lambda(x, y) = \max\left\{\frac{x + y - 1 + \lambda xy}{1 + \lambda},\ 0\right\},$$

$$\bot_\lambda(x, y) = \min\left\{x + y + \lambda xy,\ 1\right\}.$$

$\lambda = 0$ leads to $\top_{\text{Łuka}}$ and $\bot_{\text{Łuka}}$, resp.

$\lambda \to \infty$ results in $\top_{\text{prod}}$ and $\bot_{\text{sum}}$, resp.

$\lambda \to -1$ creates $\top_{-1}$ and $\bot_{-1}$, resp.

Sugeno-Weber Family II

Additive generators f_λ of $\top_\lambda$ are

$$f_\lambda(x) = \begin{cases} 1 - x & \text{if } \lambda = 0 \\ 1 - \frac{\log(1+\lambda x)}{\log(1+\lambda)} & \text{otherwise.} \end{cases}$$

$\{\top_\lambda\}_{\lambda > -1}$ are increasing functions of parameter λ.

Additive generators of $\bot_\lambda$ are $g_\lambda(x) = 1 - f_\lambda(x)$.

Yager Family

For $0 < p < \infty$ and $x, y \in [0, 1]$, define

$$\top_p(x, y) = \max\left\{1 - ((1-x)^p + (1-y)^p)^{1/p},\ 0\right\},$$
$$\bot_p(x, y) = \min\left\{(x^p + y^p)^{1/p},\ 1\right\}.$$

Additive generators of $\top_p$ are

$$f_p(x) = (1-x)^p,$$

and of $\bot_p$ are

$$g_p(x) = x^p.$$

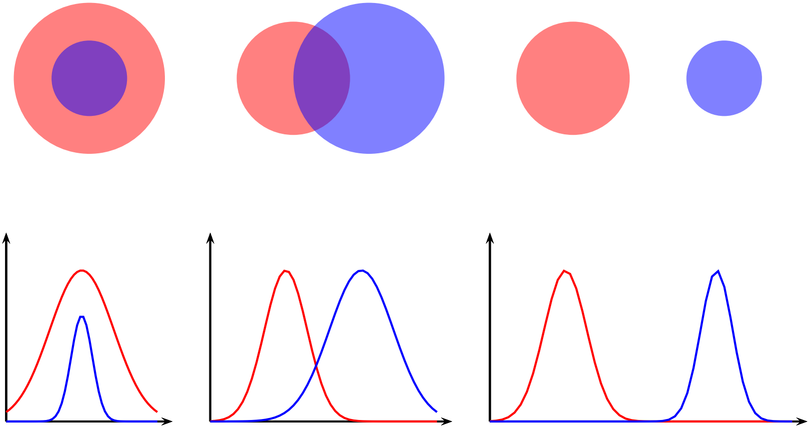
$\{\top_p\}_{0<p<\infty}$ are strictly increasing in p.

Note that $\lim_{p\to+0} \top_p = \top_{\text{Łuka}}$.

Outline

Fuzzy Implications

crisp: $x \in A \Rightarrow x \in B$, **fuzzy:** $x \in \mu \Rightarrow x \in \mu'$



Definitions of Fuzzy Implications

One way of defining I is to use $\forall a, b \in \{0, 1\}$

$$I(a, b) = \neg a \vee b.$$

In fuzzy logic, disjunction and negation are *t*-conorm and fuzzy complement, resp., thus $\forall a, b \in [0, 1]$

$$I(a, b) = \bot(\sim a, b).$$

Another way in classical logic is $\forall a, b \in \{0, 1\}$

$$I(a, b) = \max \{x \in \{0, 1\} \mid a \wedge x \leq b\}.$$

In fuzzy logic, conjunction represents *t*-norm, thus $\forall a, b \in [0, 1]$

$$I(a, b) = \sup \{x \in [0, 1] \mid \top(a, x) \leq b\}.$$

So, classical definitions are equal, fuzzy extensions are not.

Definitions of Fuzzy Implications

$I(a, b) = \bot(\sim a, b)$ may also be written as either

$$I(a, b) = \neg a \vee (a \wedge b) \qquad \text{or}$$
$$I(a, b) = (\neg a \wedge \neg b) \vee b.$$

Fuzzy logical extensions are thus, respectively,

$$I(a, b) = \bot(\sim a, \top(a, b)),$$
$$I(a, b) = \bot(\top(\sim a, \sim b), b)$$

where $(\top, \bot, n)$ must be a *De Morgan triplet*.

So again, classical definitions are equal, fuzzy extensions are not.

reason: Law of absorption of negation does not hold in fuzzy logic.

S-Implications

Implications based on $I(a, b) = \bot(\sim a, b)$ are called **S-implications**.

Symbol S is often used to denote t-conorms.

Four well-known S-implications are based on $\sim a = 1 - a$:

Name	$I(a, b)$	$\bot(a, b)$
Kleene-Dienes	$I_{\max}(a, b) = \max(1 - a, b)$	$\max(a, b)$
Reichenbach	$I_{\text{sum}}(a, b) = 1 - a + ab$	$a + b - ab$
Łukasiewicz	$I_{Ł}(a, b) = \min(1,\ 1 - a + b)$	$\min(1,\ a + b)$
largest	$I_{-1}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$	$\begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$

S-Implications

The drastic sum $\bot_{-1}$ leads to the largest S-implication I_{-1} due to the following theorem:

Theorem
Let $\bot_1, \bot_2$ be t-conorms such that $\bot_1(a, b) \leq \bot_2(a, b)$ for all $a, b \in [0, 1]$. Let I_1, I_2 be S-implications based on same fuzzy complement $\sim$ and $\bot_1, \bot_2$, respectively. Then $I_1(a, b) \leq I_2(a, b)$ for all $a, b \in [0, 1]$.

Since $\bot_{-1}$ leads to the largest S-implication, similarly, $\bot_{\max}$ leads to the smallest S-implication $I_{\max}$.

Furthermore,

$$I_{\max} \leq I_{\text{sum}} \leq I_{\text{Ł}} \leq I_{-1}.$$

R-Implications

$I(a,b) = \sup\{x \in [0,1] \mid \top(a,x) \leq b\}$ leads to R-**implications**.

Symbol R represents close connection to residuated semigroup.

Three well-known R-implications are based on $\sim a = 1 - a$:

- Standard fuzzy intersection leads to **Gödel implication**

$$I_{\min}(a,b) = \sup\{x \mid \min(a,x) \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b. \end{cases}$$

- Product leads to **Goguen implication**

$$I_{\text{prod}}(a,b) = \sup\{x \mid ax \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b. \end{cases}$$

- Łukasiewicz t-norm leads to **Łukasiewicz implication**

$$I_{Ł}(a,b) = \sup\{x \mid \max(0, a + x - 1) \leq b\} = \min(1,\ 1 - a + b).$$

R-Implications

Name	Formula	$\top(a,b) =$
Gödel	$I_{\min}(a,b) = \begin{cases} 1, & \text{if } a \le b \\ b, & \text{if } a > b \end{cases}$	$\min(a,b)$
Goguen	$I_{\text{prod}}(a,b) = \begin{cases} 1, & \text{if } a \le b \\ b/a, & \text{if } a > b \end{cases}$	ab
Łukasiewicz	$I_{Ł}(a,b) = \min(1,\ 1 - a + b)$	$\max(0,\ a + b - 1)$
largest	$I_{L}(a,b) = \begin{cases} b, & \text{if } a = 1 \\ 1, & \text{otherwise} \end{cases}$	not defined

I_L is actually the limit of all R-implications.

It serves as least upper bound.

It cannot be defined by $I(a,b) = \sup\{x \in [0,1] \mid \top(a,x) \le b\}$.

R-Implications

Theorem

Let $\top_1, \top_2$ be t-norms such that $\top_1(a,b) \leq \top_2(a,b)$ for all $a,b \in [0,1]$. Let I_1, I_2 be R-implications based on $\top_1, \top_2$, respectively. Then $I_1(a,b) \geq I_2(a,b)$ for all $a,b \in [0,1]$.

It follows that Gödel $I_{\min}$ is the smallest R-implication.

Furthermore,

$$I_{\min} \leq I_{\text{prod}} \leq I_{Ł} \leq I_{L}.$$

QL-Implications

Implications based on $I(a, b) = \bot(\sim a, \top(a, b))$ are called QL-**implications** (QL from quantum logic).

Four well-known QL-implications are based on $\sim a = 1 - a$:

- Standard min and max lead to **Zadeh implication**

$$I_Z(a, b) = \max[1 - a, \min(a, b)].$$

- The algebraic product and sum lead to

$$I_{\mathrm{p}}(a, b) = 1 - a + a^2 b.$$

- Using $\top_{Ł}$ and $\bot_{Ł}$ leads to **Kleene-Dienes implication** again.
- Using $\top_{-1}$ and $\bot_{-1}$ leads to

$$I_{\mathrm{q}}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } a \neq 1, b \neq 1 \\ 1, & \text{if } a \neq 1, b = 1. \end{cases}$$

Axioms

All I come from generalizations of the classical implication.

They collapse to the classical implication when truth values are 0 or 1.

Generalizing classical properties leads to following axioms:

- $a \leq b$ implies $I(a, x) \geq I(b, x)$ *(monotonicity in 1st argument)*
- $a \leq b$ implies $I(x, a) \leq I(x, b)$ *(monotonicity in 2nd argument)*
- $I(0, a) = 1$ *(dominance of falsity)*
- $I(1, b) = b$ *(neutrality of truth)*
- $I(a, a) = 1$ *(identity)*
- $I(a, I(b, c)) = I(b, I(a, c))$ *(exchange property)*
- $I(a, b) = 1$ if and only if $a \leq b$ *(boundary condition)*
- $I(a, b) = I(\sim b, \sim a)$ for fuzzy complement $\sim$ *(contraposition)*
- I is a continuous function *(continuity)*

Generator Function

I that satisfy all listed axioms are characterized by this theorem:

Theorem

A function $I : [0,1]^2 \to [0,1]$ *satisfies Axioms 1–9 of fuzzy implications for a particular fuzzy complement* $\sim$ *if and only if there exists a strict increasing continuous function* $f : [0,1] \to [0,\infty)$ *such that* $f(0) = 0$,

$$I(a,b) = f^{(-1)}(f(1) - f(a) + f(b))$$

for all $a, b \in [0,1]$, *and*

$$\sim a = f^{-1}(f(1) - f(a))$$

for all $a \in [0,1]$.

Example

Consider $f_\lambda(a) = \ln(1 + \lambda a)$ with $a \in [0, 1]$ and $\lambda > 0$.

Its pseudo-inverse is

$$f_\lambda^{(-1)}(a) = \begin{cases} \frac{e^a - 1}{\lambda}, & \text{if } 0 \leq a \leq \ln(1 + \lambda) \\ 1, & \text{otherwise.} \end{cases}$$

The fuzzy complement generated by f for all $a \in [0, 1]$ is

$$n_\lambda(a) = \frac{1 - a}{1 + \lambda a}.$$

The resulting fuzzy implication for all $a, b \in [0, 1]$ is thus

$$I_\lambda(a, b) = \min\left(1, \ \frac{1 - a + b + \lambda b}{1 + \lambda a}\right).$$

If $\lambda \in (-1, 0)$, then I_λ is called **pseudo-Łukasiewicz implication**.

List of Fuzzy Implications

Name	Class	Form $I(a,b) =$	Axioms	Complement
Gaines-Rescher		$\begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$	1–8	$1 - a$
Gödel	R	$\begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$	1–7	
Goguen	R	$\begin{cases} 1 & \text{if } a \leq b \\ b/a & \text{otherwise} \end{cases}$	1–7, 9	
Kleene-Dienes	S,QL	$\max(1 - a, b)$	1–4, 6, 8, 9	$1 - a$
Łukasiewicz	R, S	$\min(1, 1 - a + b)$	1–9	$1 - a$
Pseudo-Łukasiewicz 1	R, S	$\min\left[1, \frac{1-a+(1+\lambda)b}{1+\lambda a}\right]$	1–9	$\frac{1-a}{1+\lambda a}$, $(\lambda > -1)$
Pseudo-Łukasiewicz 2	R, S	$\min\left[1, 1 - a^w + b^w\right]$	1–9	$(1 - a^w)^{\frac{1}{w}}$, $(w > 0)$
Reichenbach	S	$1 - a + ab$	1–4, 6, 8, 9	$1 - a$
Wu		$\begin{cases} 1 & \text{if } a \leq b \\ \min(1 - a, b) & \text{otherwise} \end{cases}$	1–3,5,7,8	$1 - a$
Zadeh	QL	$\max[1 - a, \min(a, b)]$	1–4, 9	$1 - a$

Which Fuzzy Implication?

Since the meaning of I is not unique, we must resolve the following question:

Which I should be used for calculating the fuzzy relation R?

Hence meaningful criteria are needed.

They emerge from various fuzzy inference rules, *i.e.* modus ponens, modus tollens, hypothetical syllogism.