Outline

1. Representation of Fuzzy Sets
   - Cantor’s Theory
   - Alpha-cuts
   - Properties based on Alpha-cuts

2. Fuzzy Set Operators based on Multi-valued Logics
Definition of a “set”

“By a set we understand every collection made into a whole of definite, distinct objects of our intuition or of our thought.” (Georg Cantor).

For a set in Cantor’s sense, the following properties hold:

- \( x \neq \{x\} \).
- If \( x \in X \) and \( X \in Y \), then \( x \notin Y \).
- The Set of all subsets of \( X \) is denoted as \( 2^X \).
- \( \emptyset \) is the empty set and thus very important.
# Extension to a fuzzy set

## Ling. description vs. model

<table>
<thead>
<tr>
<th>ling. description</th>
<th>model</th>
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<tbody>
<tr>
<td>all numbers smaller than 10</td>
<td>[ \mu : X \rightarrow [0, 1] ]</td>
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<tr>
<td>all numbers almost equal to 10</td>
<td>[ \mu : X \rightarrow [0, 1] ]</td>
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**Definition**

A fuzzy set \( \mu \) of \( X \neq \emptyset \) is a function from the reference set \( X \) to the unit interval, i.e. \( \mu : X \rightarrow [0, 1] \). \( \mathcal{F}(X) \) represents the set of all fuzzy sets of \( X \), i.e. \( \mathcal{F}(X) \overset{\text{def}}{=} \{ \mu \mid \mu : X \rightarrow [0, 1] \} \).
**Vertical Representation**

So far, fuzzy sets were described by their characteristic/membership function and assigning degree of membership $\mu(x)$ to each element $x \in X$.

That is the **vertical representation** of the corresponding fuzzy set, e.g. linguistic expression like “about $m$”

$$\mu_{m,d}(x) = \begin{cases} 1 - \left| \frac{m-x}{d} \right|, & \text{if } m - d \leq x \leq m + d \\ 0, & \text{otherwise}, \end{cases}$$

or “approximately between $b$ and $c$”

$$\mu_{a,b,c,d}(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \leq x < b \\ 1, & \text{if } b \leq x \leq c \\ \frac{x-d}{c-d}, & \text{if } c < x \leq d \\ 0, & \text{if } x < a \text{ or } x > d. \end{cases}$$
Horizontal Representation

Another representation is very often applied as follows:

For all membership degrees $\alpha$ belonging to chosen subset of $[0, 1]$, human expert lists elements of $X$ that fulfill vague concept of fuzzy set with degree $\geq \alpha$.

That is the horizontal representation of fuzzy sets by their $\alpha$-cuts.

Definition

Let $\mu \in \mathcal{F}(X)$ and $\alpha \in [0, 1]$. Then the sets

$$\left[\mu\right]_{\alpha} = \{x \in X \mid \mu(x) \geq \alpha\}, \quad \overline{\left[\mu\right]_{\alpha}} = \{x \in X \mid \mu(x) > \alpha\}$$

are called the $\alpha$-cut and strict $\alpha$-cut of $\mu$. 
A Simple Example

Let $A \subseteq X$, $\chi_A : X \to [0, 1]$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}$$

Then $[\chi_A]_\alpha = A$.

$\chi_A$ is called indicator function or characteristic function of $A$. 

Let $\mu$ be triangular function on $\mathbb{R}$ as shown above.

$\alpha$-cut of $\mu$ can be constructed by

1. drawing horizontal line parallel to $x$-axis through point $(0, \alpha)$,
2. projecting this section onto $x$-axis.

$$[\mu_{\alpha}] = \begin{cases} [a + \alpha(m - a), b - \alpha(b - m)], & \text{if } 0 < \alpha \leq 1, \\ \mathbb{R}, & \text{if } \alpha = 0. \end{cases}$$
Properties of $\alpha$-cuts I

Any fuzzy set can be described by specifying its $\alpha$-cuts. That is the $\alpha$-cuts are important for application of fuzzy sets.

**Theorem**

Let $\mu \in \mathcal{F}(X)$, $\alpha \in [0, 1]$ and $\beta \in [0, 1]$.

(a) $[\mu]_0 = X$,

(b) $\alpha < \beta \implies [\mu]_\alpha \supseteq [\mu]_\beta$,

(c) $\bigcap_{\alpha : \alpha < \beta} [\mu]_\alpha = [\mu]_\beta$. 
Properties of $\alpha$-cuts II

Theorem (Representation Theorem)

Let $\mu \in \mathcal{F}(X)$. Then

$$\mu(x) = \sup_{\alpha \in [0,1]} \left\{ \min(\alpha, \chi_{[\mu]_\alpha}(x)) \right\}$$

where $\chi_{[\mu]_\alpha}(x) = \begin{cases} 1, & \text{if } x \in [\mu]_\alpha \\ 0, & \text{otherwise} \end{cases}$

So, fuzzy set can be obtained as upper envelope of its $\alpha$-cuts. Simply draw $\alpha$-cuts parallel to horizontal axis in height of $\alpha$.

In applications it is recommended to select finite subset $L \subseteq [0,1]$ of relevant degrees of membership.

They must be semantically distinguishable.

That is, fix level sets of fuzzy sets to characterize only for these levels.
System of Sets

In this manner we obtain system of sets

\[ \mathcal{A} = (A_\alpha)_{\alpha \in L}, \quad L \subseteq [0,1], \quad \text{card}(L) \in \mathbb{N}. \]

\( \mathcal{A} \) must satisfy consistency conditions for \( \alpha, \beta \in L \):

(a) \( 0 \in L \implies A_0 = X \), (fixing of reference set)

(b) \( \alpha < \beta \implies A_\alpha \supseteq A_\beta \). (monotonicity)

This induces fuzzy set

\[ \mu_\mathcal{A} : X \rightarrow [0,1], \]
\[ \mu_\mathcal{A}(x) = \sup_{\alpha \in L} \{ \min(\alpha, \chi_{A_\alpha}(x)) \}. \]

If \( L \) is not finite but comprises all values \([0,1]\), then \( \mu \) must satisfy

(c) \( \bigcap_{\alpha:\alpha<\beta} A_\alpha = A_\beta \). (condition for continuity)
Representation of Fuzzy Sets

Definition

$F\mathcal{L}(X)$ denotes the set of all families $(A_\alpha)_{\alpha \in [0,1]}$ of sets that satisfy

(a) $A_0 = X$,
(b) $\alpha < \beta \implies A_\alpha \supseteq A_\beta$,
(c) $\bigcap_{\alpha: \alpha < \beta} A_\alpha = A_\beta$.

Any family $\mathcal{A} = (A_\alpha)_{\alpha \in [0,1]}$ of sets of $X$ that satisfy (a)–(b) represents fuzzy set $\mu_\mathcal{A} \in \mathcal{F}(X)$ with

$$\mu_\mathcal{A}(x) = \sup \{ \alpha \in [0,1] \mid x \in A_\alpha \}.$$ 

Vice versa: If there is $\mu \in \mathcal{F}(X)$, then family $([\mu]_\alpha)_{\alpha \in [0,1]}$ of $\alpha$-cuts of $\mu$ satisfies (a)–(b).
“Approximately 5 or greater than or equal to 7”

An Exemplary Horizontal View

Suppose that $X = [0, 15]$.

An expert chooses $L = \{0, 0.25, 0.5, 0.75, 1\}$ and $\alpha$-cuts:

- $A_0 = [0, 15]$,
- $A_{0.25} = [3, 15]$,
- $A_{0.5} = [4, 6] \cup [7, 15]$,
- $A_{0.75} = [4.5, 5.5] \cup [7, 15]$,
- $A_1 = \{5\} \cup [7, 15]$.

The family $(A_\alpha)_{\alpha \in L}$ of sets induces upper shown fuzzy set.
“Approximately 5 or greater than or equal to 7”
An Exemplary Vertical View

$\mu_A$ is obtained as upper envelope of the family $A$ of sets.

The difference between horizontal and vertical view is obvious:

The horizontal representation is easier to process in computers.
Also, restricting the domain of $x$-axis to a discrete set is usually done.
Horizontal Representation in the Computer

Fuzzy sets are usually stored as chain of linear lists.

For each $\alpha$-level, $\alpha \neq 0$.

A finite union of closed intervals is stored by their bounds.

This data structure is appropriate for arithmetic operators.
Support and Core of a Fuzzy Set

Definition
The support $S(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the crisp set that contains all elements of $X$ that have nonzero membership. Formally

$$S(\mu) = [\mu]_0 = \{x \in X \mid \mu(x) > 0\}.$$

Definition
The core $C(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the crisp set that contains all elements of $X$ that have membership of one. Formally,

$$C(\mu) = [\mu]_1 = \{x \in X \mid \mu(x) = 1\}.$$
Height of a Fuzzy Set

Definition
The *height* $h(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the largest membership grade obtained by any element in that set. Formally,

$$h(\mu) = \sup_{x \in X} \mu(x).$$

$h(\mu)$ may also be viewed as supremum of $\alpha$ for which $[\mu]_\alpha \neq \emptyset$.

Definition
A fuzzy set $\mu$ is called *normal* when $h(\mu) = 1$. It is called *subnormal* when $h(\mu) < 1$. 
Convex Fuzzy Sets I

Definition
Let $X$ be a vector space. A fuzzy set $\mu \in \mathcal{F}(X)$ is called fuzzy convex if its $\alpha$-cuts are convex for all $\alpha \in (0, 1]$.

The membership function of a convex fuzzy set is not a convex function.

The classical definition: The membership functions are actually concave.
Fuzzy Numbers

Definition

\( \mu \) is a fuzzy number if and only if \( \mu \) is normal and \([\mu]_\alpha\) is bounded, closed, and convex \( \forall \alpha \in (0, 1] \).

Example:

The term *approximately* \( x_0 \) is often described by a parametrized class of membership functions, e.g.

\[
\mu_1(x) = \max\{0, 1 - c_1|x - x_0|\}, \quad c_1 > 0, \\
\mu_2(x) = \exp(-c_2\|x - x_0\|_p), \quad c_2 > 0, \quad p \geq 1.
\]
Theorem

A fuzzy set $\mu \in \mathcal{F}(\mathbb{R})$ is convex if and only if

$$\mu(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu(x_1), \mu(x_2)\}$$

for all $x_1, x_2 \in \mathbb{R}$ and all $\lambda \in [0, 1]$. 
Fuzzy Numbers – Example

Upper semi-continuous functions are most convenient.

In many applications (e.g. fuzzy control) the class of the functions and their exact parameters have a limited influence on the results.

Only local monotonicity of the functions is really necessary.

In other applications (e.g. medical diagnosis) more precise membership degrees are needed.

\[
[\mu]_\alpha = \begin{cases} 
[1, 2] & \text{if } \alpha \geq 0.5, \\
[0.5 + \alpha, 2) & \text{if } 0 < \alpha < 0.5, \\
\mathbb{R} & \text{if } \alpha = 0 
\end{cases}
\]
Set Operators...

...are defined by using traditional logics operator

Let $X$ be universe of discourse (universal set):

$$A \cap B = \{x \in X \mid x \in A \land x \in B\}$$

$$A \cup B = \{x \in X \mid x \in A \lor x \in B\}$$

$$A^c = \{x \in X \mid x \notin A\} = \{x \in X \mid \neg(x \in A)\}$$

$$A \subseteq B \text{ if and only if } (x \in A) \rightarrow (x \in B) \text{ for all } x \in X$$

One idea to define fuzzy set operators: use fuzzy logics.
The Traditional or Aristotlelian Logic
What is logic about? Different schools speak different languages!

There are traditional, linguistic, psychological, epistemological and mathematical schools.

Traditional logic has been founded by Aristotle (384-322 B.C.).

Aristotlelian logic can be seen as a formal approach to human reasoning. It's still used today in Artificial Intelligence for knowledge representation and reasoning about knowledge.

Detail of “The School of Athens” by R. Sanzio (1509) showing Plato (left) and his student Aristotle (right).
Outline

1. Representation of Fuzzy Sets

2. Fuzzy Set Operators based on Multi-valued Logics
   - Boolean Algebra
   - n-valued Logics
   - Fuzzy Logic
Classical Logic: An Overview

Logic studies methods/principles of reasoning.
Classical logic deals with propositions (either true or false).

The propositional logic handles combination of logical variables.
Key idea: how to express n-ary logic functions with logic primitives, e.g. ¬, ∧, ∨, →.

A set of logic primitives is complete if any logic function can be composed by a finite number of these primitives, e.g. {¬, ∧, ∨}, {¬, ∧}, {¬, →}, {↓} (NOR), {||} (NAND) (this was also discussed during the 1st exercise).
Inference Rules

When a variable represented by logical formula is: true for all possible truth values, i.e. it is called tautology, false for all possible truth values, i.e. it is called contradiction.

Various forms of tautologies exist to perform deductive inference. They are called inference rules:

\[
\begin{align*}
(a \land (a \rightarrow b)) & \rightarrow b \quad \text{(modus ponens)} \\
(\neg b \land (a \rightarrow b)) & \rightarrow \neg a \quad \text{(modus tollens)} \\
((a \rightarrow b) \land (b \rightarrow c)) & \rightarrow (a \rightarrow c) \quad \text{(hypothetical syllogism)}
\end{align*}
\]

e.g. modus ponens: given two true propositions \(a\) and \(a \rightarrow b\) (premises), truth of proposition \(b\) (conclusion) can be inferred.

Every tautology remains a tautology when any of its variables is replaced with an arbitrary logic formula.
Boolean Algebra

The propositional logic based on finite set of logic variables is isomorphic to finite set theory.

Both of these systems are isomorphic to a finite Boolean algebra.

Definition
A Boolean algebra on a set $B$ is defined as quadruple $\mathcal{B} = (B, +, \cdot, \overline{\cdot})$ where $B$ has at least two elements (bounds) 0 and 1, $+$ and $\cdot$ are binary operators on $B$, and $\overline{\cdot}$ is a unary operator on $B$ for which the following properties hold.
Properties of Boolean Algebras I

(B1) Idempotence
\[ a + a = a \quad a \cdot a = a \]

(B2) Commutativity
\[ a + b = b + a \quad a \cdot b = b \cdot a \]

(B3) Associativity
\[ (a + b) + c = a + (b + c) \quad (a \cdot b) \cdot c = a \cdot (b \cdot c) \]

(B4) Absorption
\[ a + (a \cdot b) = a \quad a \cdot (a + b) = a \]

(B5) Distributivity
\[ a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad a + (b \cdot c) = (a + b) \cdot (a + c) \]

(B6) Universal Bounds
\[ a + 0 = a, \quad a + 1 = 1 \quad a \cdot 1 = a, \quad a \cdot 0 = 0 \]

(B7) Complementary
\[ a + \overline{a} = 1 \quad a \cdot \overline{a} = 0 \]

(B8) Involution
\[ \overline{\overline{a}} = a \quad \overline{a + b} = \overline{a} \cdot \overline{b} \]

(B9) Dualization
\[ \overline{a \cdot b} = \overline{a} + \overline{b} \]

Properties (B1)-(B4) are common to every lattice,

i.e. a Boolean algebra is a distributive (B5), bounded (B6), and complemented (B7)-(B9) lattice,

i.e. every Boolean algebra can be characterized by a partial ordering on a set, i.e. \( a \leq b \) if \( a \cdot b = a \) or, alternatively, if \( a + b = b \).
Set Theory, Boolean Algebra, Propositional Logic

Every theorem in one theory has a counterpart in each other theory. Counterparts can be obtained applying the following substitutions:

<table>
<thead>
<tr>
<th>Meaning</th>
<th>Set Theory</th>
<th>Boolean Algebra</th>
<th>Prop. Logic</th>
</tr>
</thead>
<tbody>
<tr>
<td>values</td>
<td>$2^X$</td>
<td>$B$</td>
<td>$\mathcal{L}(V)$</td>
</tr>
<tr>
<td>“meet”/“and”</td>
<td>$\cap$</td>
<td>$\cdot$</td>
<td>$\land$</td>
</tr>
<tr>
<td>“join”/“or”</td>
<td>$\cup$</td>
<td>$+$</td>
<td>$\lor$</td>
</tr>
<tr>
<td>“complement”/“not”</td>
<td>$c$</td>
<td>$-$</td>
<td>$\neg$</td>
</tr>
<tr>
<td>identity element</td>
<td>$X$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>zero element</td>
<td>$\emptyset$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>partial order</td>
<td>$\subseteq$</td>
<td>$\leq$</td>
<td>$\rightarrow$</td>
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</table>

power set $2^X$, set of logic variables $V$, set of all combinations $\mathcal{L}(V)$ of truth values of $V$
The Basic Principle of Classical Logic

The Principle of Bivalence:
“Every proposition is either true or false.”
It has been formally developed by Tarski.

Łukasiewicz suggested to replace it by
The Principle of Valence:
“Every proposition has a truth value.”
Propositions can have intermediate truth value,
expressed by a number from the unit interval \([0, 1]\).
Aristotle introduced a logic of terms and drawing conclusion from two premises.

The great Greeks (Chrisippus) also developed logic of propositions.

Jan Łukasiewicz founded the multi-valued logic.

The multi-valued logic is to fuzzy set theory what classical logic is to set theory.
Three-valued Logics

A 2-valued logic can be extended to a 3-valued logic *in several ways*, *i.e.* different three-valued logics have been well established:

truth, falsity, indeterminacy are denoted by 1, 0, and 1/2, *resp.*

The negation \( \neg a \) is defined as \( 1 - a \), *i.e.* \( \neg 1 = 0, \neg 0 = 1 \) and \( \neg 1/2 = 1/2 \).

Other primitives, *e.g.* \( \land, \lor, \to, \leftrightarrow \), differ from logic to logic.

Five well-known three-valued logics (named after their originators) are defined in the following.
Primitives of Some Three-valued Logics

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>Łukasiewicz</th>
<th>Bochvar</th>
<th>Kleene</th>
<th>Heyting</th>
<th>Reichenbach</th>
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All of them fully conform the usual definitions for $a, b \in \{0, 1\}$. They differ from each other only in their treatment of $1/2$.

**Question:** Do they satisfy the law of contradiction ($a \land \neg a = 0$) and the law of excluded middle ($a \lor \neg a = 1$)?
n-valued Logics

After the three-valued logics: generalizations to $n$-valued logics for arbitrary number of truth values $n \geq 2$.

In the 1930s, various $n$-valued logics were developed.

Usually truth values are assigned by rational number in $[0, 1]$.

Key idea: uniformly divide $[0, 1]$ into $n$ truth values.

**Definition**

The set $T_n$ of truth values of an $n$-valued logic is defined as

$$T_n = \left\{ 0 = \frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, \frac{n-1}{n-1} = 1 \right\}.$$

These values can be interpreted as degree of truth.
Primitives in \( n \)-valued Logics

Łukasiewicz proposed first series of \( n \)-valued logics for \( n \geq 2 \).

In the early 1930s, he simply generalized his three-valued logic.

It uses truth values in \( T_n \) and defines primitives as follows:

\[
\begin{align*}
\neg a &= 1 - a \\
\land a \land b &= \min(a, b) \\
\lor a \lor b &= \max(a, b) \\
\rightarrow a \rightarrow b &= \min(1, 1 + b - a) \\
\leftrightarrow a \leftrightarrow b &= 1 - |a - b|
\end{align*}
\]

The \( n \)-valued logic of Łukasiewicz is denoted by \( L_n \).

The sequence \((L_2, L_3, \ldots, L_\infty)\) contains the classical two-valued logic \( L_2 \) and an infinite-valued logic \( L_\infty \) (rational countable values \( T_\infty \)).

The infinite-valued logic \( L_1 \) (standard Łukasiewicz logic) is the logic with all real numbers in \([0, 1]\) \((1 = \text{cardinality of continuum})\).
Zadeh’s fuzzy logic proposal was much simpler

In 1965, he proposed a logic with values in $[0, 1]$:

$$\neg a = 1 - a,$$
$$a \land b = \min(a, b),$$
$$a \lor b = \max(a, b).$$

The set operators are defined pointwise as follows for $\mu, \mu'$:

$$\neg \mu : X \rightarrow X, \neg \mu(x) = 1 - \mu(x),$$
$$\mu \land \mu' : X \rightarrow X(\mu \land \mu')(x) = \min\{\mu(x), \mu'(x)\},$$
$$\mu \lor \mu' : X \rightarrow X(\mu \lor \mu')(x) = \max\{\mu(x), \mu'(x)\}.$$
Standard Fuzzy Set Operators – Example

- Fuzzy complement
- Fuzzy intersection
- Two fuzzy sets
- Fuzzy union
Is Zadeh’s logic a Boolean algebra?

**Theorem**

\(( \mathcal{F}(X), \land, \lor, \lnot) \) is a complete distributive lattice but no Boolean algebra.

**Proof.**

Consider \( \mu : X \to X \) with \( x \mapsto 0.5 \), then \( \lnot \mu(x) = 0.5 \) for all \( x \) and \( \mu \land \lnot \mu \neq \chi_\emptyset \). □