Assignment Sheet 3

Assignment 9 Lattices/Boolean Algebras

The transfer from logic to set theory is possible because both systems have basically the same structure. This structure is captured by the algebraic notion of a Boolean algebra. A Boolean algebra on a set B is defined as quadruple $\mathcal{B} = (B, +, \cdot, \bar{\ })$ where B has at least two elements (bounds), *i.e.* 0, 1, and $+, \cdot : B \times B \to B$ are binary operations on B, and $\bar{\ } : B \to B$ is a unary operation on B for which the following axioms hold for all $a, b, c \in B$:

1)	$(a + b) + c = a + (b + c)$,	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$	(associativity)
2)	$a + b = b + a$,	$a \cdot b = b \cdot a$	(commutativity)
3)	$(a + b) \cdot a = a$,	$(a \cdot b) + a = a$	(absorption)
4)	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$,	$a + (b \cdot c) = (a + b) \cdot (a + c)$	(distributivity)
5)	$a + (b \cdot \bar{b}) = a$,	$a \cdot (b + \bar{b}) = a$	

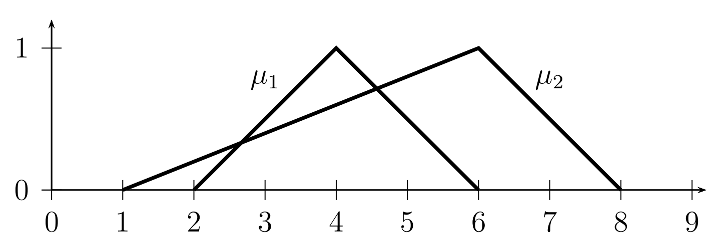
If only the first three axioms are satisfied, the structure is called a lattice. If the first four are satisfied, it is called a distributive lattice.

Show that the set of fuzzy truth values (the real interval $[0, 1]$) together with the standard fuzzy operations $\top(a, b) = \min\{a, b\}$ (conjunction), $\bot(a, b) = \max\{a, b\}$ (disjunction) and $\sim a = 1 - a$ (negation) is a distributive lattice but not a Boolean algebra.

Assignment 10 α-cuts

Compute the sets of α-cuts for both

a) the two fuzzy sets μ_1 and μ_2 given by their graphs as follows



and

b) the fuzzy set defined as follows

$$\mu(x) = \begin{cases} 1 - (x - 2)^2, & \text{if } 1 \le x \le 3 \\ 0, & \text{otherwise.} \end{cases}$$

Assignment 11 Representation of Fuzzy Sets

Let $(A_\alpha)_{\alpha\in[0,1]}$ be the system of sets defined by

$$A_\alpha = \begin{cases} \left[1-\sqrt{\ln\frac{1}{\alpha}}, 1+\sqrt{\ln\frac{1}{\alpha}}\right], & \text{if } \alpha > 0 \\ \mathbb{R}, & \text{if } \alpha = 0. \end{cases}$$

a) Show that this system of sets satisfies the conditions that are satisfied by the set of α-cuts of a fuzzy set (as stated in a theorem of the lecture), *i.e.*

   (i) $[\mu]_0 = U$, where $U = \mathbb{R}$ in this case,

   (ii) $\forall\alpha,\beta: \quad \alpha \leq \beta \Rightarrow [\mu]_\alpha \supseteq [\mu]_\beta$,

   (iii) $\forall\beta\in[0,1]: \quad \bigcap_{\alpha:\alpha<\beta} [\mu]_\alpha = [\mu]_\beta$.

b) Find the membership function μ of the fuzzy set that corresponds to this system of sets.

Assignment 12 Programming

Implement the data structure from the lecture that can horizontally represent a fuzzy set by its α-cuts. In detail, solve the following subtasks.

a) Enable the user to enter a finite subset $L \subseteq [0,1]$ of relevant degrees of membership.

b) For every $l \in L$, the user shall be able to specify the α-cuts corresponding to l.

c) Finally, implement a method that returns the membership degree $\mu(x)$ of an element x given your data structure of a fuzzy set.