Nonstandard Concepts for Handling Imprecise Data and Imprecise Probabilities
Problems with Probability Theory

Representation of Ignorance

We are given a die with faces 1, \ldots, 6
What is the certainty of showing up face \( i \) ?

- Conduct a statistical survey (roll the die 10000 times) and estimate the relative frequency: \( P(\{i\}) = \frac{1}{6} \)
- Use subjective probabilities (which is often the normal case): We do not know anything (especially and explicitly we do not have any reason to assign unequal probabilities), so the most plausible distribution is a uniform one.

Problem: Uniform distribution because of ignorance or extensive statistical tests

Experts analyze aircraft shapes: 3 aircraft types \( A, B, C \)
“It is type \( A \) or \( B \) with 90\% certainty. About \( C \), I don’t have any clue and I do not want to commit myself. No preferences for \( A \) or \( B \).”

Problem: Ignorance hard to handle with Bayesian theory
“$A \subseteq X$ being an imprecise date” means: the true value $x_0$ lies in $A$ but there are no preferences on $A$.

$\Omega$ set of possible elementary events  
$\Theta = \{\xi\}$ set of observers  
$\lambda(\xi)$ importance of observer $\xi$

Some elementary event from $\Omega$ occurs and every observer $\xi \in O$ shall announce which elementary events she personally considers possible. This set is denoted by $\Gamma(\xi) \subseteq \Omega$. $\Gamma(\xi)$ is then an imprecise date.

$\lambda : 2^\Theta \rightarrow [0, 1]$ probability measure  
(interpreted as importance measure)  
$(\Theta, 2^\Theta, \lambda)$ probability space  
$\Gamma : \Theta \rightarrow 2^\Omega$ set-valued mapping
Imprecise Data (2)

Let \( A \subseteq \Omega \):

a) \( \Gamma^*(A) \overset{\text{Def}}{=} \{ \xi \in \Theta \mid \Gamma(\xi) \cap A \neq \emptyset \} \)

b) \( \Gamma_*(A) \overset{\text{Def}}{=} \{ \xi \in \Theta \mid \Gamma(\xi) \neq \emptyset \text{ and } \Gamma(\xi) \subseteq A \} \)

Remarks:

a) If \( \xi \in \Gamma^*(A) \), then it is \textit{plausible} for \( \xi \) that the occurred elementary event lies in \( A \).

b) If \( \xi \in \Gamma_*(A) \), then it is \textit{certain} for \( \xi \) that the event lies in \( A \).

c) \( \{ \xi \mid \Gamma(\xi) \neq \emptyset \} = \Gamma^*(\Omega) = \Gamma_*(\Omega) \)

Let \( \lambda(\Gamma^*(\Omega)) > 0 \). Then we call

\[
P^*(A) = \frac{\lambda(\Gamma^*(A))}{\lambda(\Gamma^*(\Omega))} \text{ the upper, and } \quad P_*(A) = \frac{\lambda(\Gamma_*(A))}{\lambda(\Gamma_*(\Omega))} \text{ the lower}
\]

probability w. r. t. \( \lambda \) and \( \Gamma \).
Example

$$\Theta = \{a, b, c, d\}$$
$$\Omega = \{1, 2, 3\}$$
$$\Gamma^*(\Omega) = \{a, b, d\}$$
$$\lambda(\Gamma^*(\Omega)) = \frac{4}{6}$$

<table>
<thead>
<tr>
<th>A</th>
<th>$$\Gamma^*(A)$$</th>
<th>$$\Gamma_*(A)$$</th>
<th>$$P^*(A)$$</th>
<th>$$P_*(A)$$</th>
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<tbody>
<tr>
<td>\emptyset</td>
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<tr>
<td>{2}</td>
<td>{b, d}</td>
<td>{b}</td>
<td>\frac{3}{4}</td>
<td>\frac{1}{4}</td>
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<tr>
<td>{3}</td>
<td>{d}</td>
<td>\emptyset</td>
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<td>{1, 2}</td>
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<td>{1, 2, 3}</td>
<td>{a, b, d}</td>
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One can consider $$P^*(A)$$ and $$P_*(A)$$ as upper and lower probability bounds.
Imprecise Data (3)

Some properties of probability bounds:

a) $P^*: 2^\Omega \rightarrow [0, 1]$

b) $0 \leq P_* \leq P^* \leq 1$, \quad $P_*(\emptyset) = P^*(\emptyset) = 0$, \quad $P_*(\Omega) = P^*(\Omega) = 1$

c) $A \subseteq B \implies P^*(A) \leq P^*(B)$ \quad and \quad $P_*(A) \leq P_*(B)$

d) $A \cap B = \emptyset \not\Rightarrow P^*(A) + P^*(B) = P^*(A \cup B)$

e) $P_*(A \cup B) \geq P_*(A) + P_*(B) - P_*(A \cap B)$

f) $P^*(A \cup B) \leq P^*(A) + P^*(B) - P^*(A \cap B)$

g) $P_*(A) = 1 - P^*(\Omega \setminus A)$
One can prove the following generalized equation:

\[ P_*(\bigcup_{i=1}^{n} A_i) \geq \sum_{\emptyset \neq I: I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \cdot P_*(\bigcap_{i \in I} A_i) \]

These set functions also play an important role in theoretical physics (capacities, Choquet, 1955). Shafer did generalize these thoughts and developed a theory of belief functions.
How is new knowledge incorporated?

Every observer announces the location of the ship in form of a subset of all possible ship locations. Given these set-valued mappings, we can derive upper and lower probabilities with the help of the observer importance measure. Let us assume the ship is certainly at sea.

How do the upper/lower probabilities change?
a) \textit{Geometric Conditioning}
( observers that give partial or full wrong information are discarded )

\[
P_*(A \mid B) = \frac{\lambda(\{\xi \in \Theta \mid \Gamma(\xi) \subseteq A \text{ and } \Gamma(\xi) \subseteq B\})}{\lambda(\{\xi \in \Theta \mid \Gamma(\xi) \subseteq B\})} = \frac{P_*(A \cap B)}{P_*(B)}
\]

\[
P^*(A \mid B) = \frac{\lambda(\{\xi \in \Theta \mid \Gamma(\xi) \subseteq B \text{ and } \Gamma(\xi) \cap A \neq \emptyset\})}{\lambda(\{\xi \in \Theta \mid \Gamma(\xi) \subseteq B\})} = \frac{P^*(A \cup \overline{B}) - P^*(\overline{B})}{1 - P^*(\overline{B})}
\]
b) **Data Revision**  
(the observed data is modified such that they fit the certain information)

\[
(P_\ast)_B(A) = \frac{P_\ast(A \cup \overline{B}) - P_\ast(\overline{B})}{1 - P_\ast(B)}
\]

\[
(P^*)_B(A) = \frac{P^*(A \cap B)}{P^*(B)}
\]

These two concepts have different semantics. There are several more belief revision concepts.
Combination of Random Sets

Let \((\Omega, 2^\Omega)\) be a space of events. Further be \((O_1, 2^{O_1}, \lambda_1)\) and \((O_2, 2^{O_2}, \lambda_2)\) spaces of independent observers.

We call \((O_1 \times O_2, \lambda_1 \cdot \lambda_2)\) the product space of observers and

\[
\Gamma : O_1 \times O_2 \rightarrow 2^\Omega, \Gamma(x_1, x_2) = \Gamma_1(x_1) \cap \Gamma_2(x_2)
\]

the combined observer function.

We obtain with

\[
(P_L)^*(A) = \frac{(\lambda_1 \cdot \lambda_2)(\{(x_1, x_2) \mid \Gamma(x_1, x_2) \neq \emptyset \land \Gamma(x_1, x_2) \subseteq A\})}{(\lambda_1 \cdot \lambda_2)(\{(x_1, x_2) \mid \Gamma(x_1, x_2) \neq \emptyset\})}
\]

the lower probability of \(A\) that respects both observations.
\( \Omega = \{1, 2, 3\} \)
\[\lambda_1: \{a\} \mapsto \frac{1}{3}, \quad \{b\} \mapsto \frac{2}{3} \]
\[\lambda_2: \{c\} \mapsto \frac{1}{2}, \quad \{d\} \mapsto \frac{1}{2} \]
\[O_1 = \{a, b\} \quad \Gamma_1: \ a \mapsto \{1, 2\}, \quad b \mapsto \{2, 3\} \]
\[O_2 = \{c, d\} \quad \Gamma_2: \ c \mapsto \{1\}, \quad d \mapsto \{2, 3\} \]

Combination:
\[O_1 \times O_2 = \{ac, bc, ad, bd\} \]
\[\lambda: \{ac\} \mapsto \frac{1}{6}, \quad \Gamma: \ ac \mapsto \{1\} \quad \Gamma^*(\Omega) = \{(x_1, x_2) \mid \Gamma(x_1, x_2) \neq \emptyset\} = \{ac, ad, bd\} \]
\[\{ad\} \mapsto \frac{1}{6}, \quad \Gamma: \ ad \mapsto \{2\} \]
\[\{bc\} \mapsto \frac{2}{6}, \quad \Gamma: \ bc \mapsto \emptyset \]
\[\{bd\} \mapsto \frac{2}{6}, \quad \Gamma: \ bd \mapsto \{2, 3\} \]
\[\lambda(\Gamma^*(\Omega)) = \frac{4}{6} \]
### Example (2)

<table>
<thead>
<tr>
<th>$A$</th>
<th>$(P_*)_{\Gamma_1}(A)$</th>
<th>$(P_*)_{\Gamma_2}(A)$</th>
<th>$(P_*)_{\Gamma}(A)$</th>
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</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
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<td>0</td>
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<td>${2, 3}$</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{4}$</td>
</tr>
<tr>
<td>${1, 2, 3}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
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Belief Functions

Motivation

\((\Theta, Q)\) Sensors

\(\Omega\) possible results, \(\Gamma : \Theta \rightarrow 2^\Omega\)

\(P_* : A \mapsto \sum_{B : B \subseteq A} m(B)\) Lower probability (Belief)

\(P^* : A \mapsto \sum_{B : B \cap A \neq \emptyset} m(B)\) Upper probability (Plausibility)

\(m : A \mapsto Q(\{\theta \in \Theta \mid \Gamma(\theta) = A\})\) mass distribution

Random sets: Dempster (1968)
Belief functions: Shafer (1974)
Development of a completely new uncertainty calculus as an alternative to Probability Theory
The function Bel : $2^\Omega \rightarrow [0, 1]$ is called belief function, if it possesses the following properties:

\[
\begin{align*}
\text{Bel}(\emptyset) &= 0 \\
\text{Bel}(\Omega) &= 1 \\
\forall n \in \mathbb{N} : \forall A_1, \ldots, A_n \in 2^\Omega : \\
\text{Bel}(A_1 \cup \cdots \cup A_n) &\geq \sum_{\emptyset \neq I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} \cdot \text{Bel}(\bigcap_{i \in I} A_i)
\end{align*}
\]

If Bel is a belief function then for $m : 2^\Omega \rightarrow \mathbb{R}$ with

\[
m(A) = \sum_{B : B \subseteq A} (-1)^{|A \setminus B|} \cdot \text{Bel}(B)
\]

the following properties hold:

\[
\begin{align*}
0 &\leq m(A) \leq 1 \\
m(\emptyset) &= 0 \\
\sum_{A \subseteq \Omega} m(A) &= 1
\end{align*}
\]
Let $|\Omega| < \infty$ and $f, g : 2^\Omega \rightarrow [0, 1]$.

\[
\forall A \subseteq \Omega : (f(A) = \sum_{B : B \subseteq A} g(B))
\]

\[\Leftrightarrow\]

\[
\forall A \subseteq \Omega : (g(A) = \sum_{B : B \subseteq A} (-1)^{|A \setminus B|} \cdot f(B))
\]

($g$ is called the Möbius transformed of $f$)

The mapping $m : 2^\Omega \rightarrow [0, 1]$ is called a mass distribution, if the following properties hold:

\[m(\emptyset) = 0\]

\[\sum_{A \subseteq \Omega} m(A) = 1\]
### Example

<table>
<thead>
<tr>
<th>$A$</th>
<th>$\emptyset$</th>
<th>${1}$</th>
<th>${2}$</th>
<th>${3}$</th>
<th>${1, 2}$</th>
<th>${2, 3}$</th>
<th>${1, 3}$</th>
<th>${1, 2, 3}$</th>
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</thead>
<tbody>
<tr>
<td>$m(A)$</td>
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<td>$\frac{1}{4}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{2}{4}$</td>
<td>0</td>
</tr>
<tr>
<td>Bel($A$)</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>0</td>
<td>$\frac{2}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{3}{4}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Belief $\equiv$ lower probability with modified semantic

\[
\text{Bel}(\{1, 3\}) = m(\emptyset) + m(\{1\}) + m(\{3\}) + m(\{1, 3\})
\]

\[
m(\{1, 3\}) = \text{Bel}(\{1, 3\}) - \text{Bel}(\{1\}) - \text{Bel}(\{3\})
\]

- $m(A)$ measure of the trust/belief that exactly $A$ occurs
- Bel$_m(A)$ measure of total belief that $A$ occurs
- Pl$_m(A)$ measure of not being able to disprove $A$ (plausibility)

\[
\text{Pl}_m(A) = \sum_{B : A \cap B \neq \emptyset} m(B) = 1 - \text{Bel}(\overline{A})
\]

Given one of $m$, Bel or Pl, the other two can be efficiently computed.
Knowledge Representation

\[ m(\Omega) = 1, \ m(A) = 0 \text{ else} \quad \text{total ignorance} \]
\[ m(\{\omega_0\}) = 1, \ m(A) = 0 \text{ else} \quad \text{value (}\omega_0\text{) known} \]
\[ m(\{\omega_i\}) = p_i, \sum_{i=1}^{n} p_i = 1 \quad \text{Bayesian analysis} \]

Further kinds of partial ignorance can be modeled.
Data Revision:

- Mass of $A$ flows onto $A \cap B$.
- Masses are normalized to 1 ($\emptyset$-mass is destroyed)

Geometric Conditioning:

- Masses that do not lie completely inside $B$, flow off
- Normalize

The mass flow can be described by specialization matrices
Motivation: Combination of $m_1$ and $m_2$

$m_1(A_i) \cdot m_2(B_j)$: Mass attached to $A_i \cap B_j$, if only $A_i$ or $B_j$ are concerned

$\sum_{i,j: A_i \cap B_j = A} m_1(A_i) \cdot m_2(B_j)$: Mass attached to $A$ (after combination)

This consideration only leads to a mass distribution, if $\sum_{i,j: A_i \cap B_j = \emptyset} m_1(A_i) \cdot m_2(B_j) = 0$.

If this sum is $> 0$ normalization takes place.
If \( m_1 \) and \( m_2 \) are mass distributions over \( \Omega \) with belief functions \( \text{Bel}_1 \) and \( \text{Bel}_2 \) and does further hold \( \sum_{i,j: A_i \cap B_j = \emptyset} m_1(A_i) \cdot m_2(B_j) < 1 \), then the function \( m : 2^\Omega \rightarrow [0, 1], m(\emptyset) = 0 \)

\[
m(A) = \frac{\sum_{B,C: B \cap C = A} m_1(B) \cdot m_2(C)}{1 - \sum_{B,C: B \cap C = \emptyset} m_1(B) \cdot m_2(C)}
\]

is a mass distribution. The belief function of \( m \) is denoted as \( \text{comb}(\text{Bel}_1, \text{Bel}_2) \) or \( \text{Bel}_1 \oplus \text{Bel}_2 \). The above formula is called the combination rule.
Example

\[ m_1(\{1, 2\}) = \frac{1}{3} \]
\[ m_1(\{2, 3\}) = \frac{2}{3} \]
\[ m_2(\{1\}) = \frac{1}{2} \]
\[ m_2(\{2, 3\}) = \frac{1}{2} \]

\[ m = m_1 \oplus m_2 : \]
\[ \{1\} \mapsto \frac{1}{6} \]
\[ \{2\} \mapsto \frac{1}{6} \]
\[ \emptyset \mapsto 0 \]
\[ \{2, 3\} \mapsto \frac{2}{6} = \frac{1}{2} \]
Combination Rule (2)

Remarks:

a) The result from the combination rule and the analysis of random sets is identical
b) There are more efficient ways of combination
c) \( \text{Bel}_1 \oplus \text{Bel}_2 = \text{Bel}_2 \oplus \text{Bel}_1 \)
d) \( \oplus \) is associative
e) \( \text{Bel}_1 \oplus \text{Bel}_1 \neq \text{Bel}_1 \) (in general)
f) \( \text{Bel}_2 : 2^\Omega \rightarrow [0, 1], \ m_2(B) = 1 \)

\[
\text{Bel}_2(A) = \begin{cases} 
1 & \text{if} \ B \subseteq A \\
0 & \text{otherwise}
\end{cases}
\]

The combination of \( \text{Bel}_1 \) and \( \text{Bel}_2 \) yields the data revision of \( m_1 \) with \( B \).
The **pignistic transformation** \( Bet \) transforms a normalized mass function \( m \) into a probability measure \( P_m = Bet(m) \) as follows:

\[
P_m(A) = \sum_{\emptyset \neq B \subseteq \Omega} m(B) \frac{|A \cap B|}{|B|}, \forall A \subseteq \Omega.
\]

It can be shown that

\[
bel(A) \leq P_m(A) \leq pl(A)
\]
There are three possible murders

Let $m(\{John\}) = 0.48$, $m(\{John, Mary\}) = 0.12$, 
$m(\{Peter, John\}) = 0.32$, $m(\Omega) = 0.08$

We have:

$$P_m(\{John\}) = 0.48 + \frac{0.12}{2} + \frac{0.32}{2} + \frac{0.08}{3} \approx 0.73$$

$$P_m(\{Peter\}) = \frac{0.32}{2} + \frac{0.08}{3} \approx 0.19$$

$$P_m(\{Mary\}) = \frac{0.12}{2} + \frac{0.08}{3} \approx 0.09$$

The picmistic transformation gives a reasonable ”Ranking”
Imprecise Probabilities

Let $x_0$ be the true value but assume there is no information about $P(A)$ to decide whether $x_0 \in A$. There are only probability boundaries.

Let $\mathcal{L}$ be a set of probability measures. Then we call

$$
(P_\mathcal{L})_* : 2^\Omega \rightarrow [0, 1], A \mapsto \inf \{P(A) \mid P \in \mathcal{L}\}
$$
the lower and

$$
(P_\mathcal{L})^* : 2^\Omega \rightarrow [0, 1], A \mapsto \sup \{P(A) \mid P \in \mathcal{L}\}
$$
the upper probability of $A$ w.r.t. $\mathcal{L}$.

a) $(P_\mathcal{L})_*(\emptyset) = (P_\mathcal{L})^*(\emptyset) = 0$; $(P_\mathcal{L})_*(\Omega) = (P_\mathcal{L})^*(\Omega) = 1$

b) $0 \leq (P_\mathcal{L})_*(A) \leq (P_\mathcal{L})^*(A) \leq 1$

c) $(P_\mathcal{L})^*(A) = 1 - (P_\mathcal{L})_*(\overline{A})$

d) $(P_\mathcal{L})_*(A) + (P_\mathcal{L})^*(B) \leq (P_\mathcal{L})^*(A \cup B)$

e) $(P_\mathcal{L})_*(A \cap B) + (P_\mathcal{L})^*(A \cup B) \not\geq (P_\mathcal{L})^*(A) + (P_\mathcal{L})^*(B)$
Belief Revision

Let $B \subseteq \Omega$ and $\mathcal{L}$ a class of probabilities. The we call

$$A \subseteq \Omega : (P_{\mathcal{L}})^*(A \mid B) = \inf\{P(A \mid B) \mid P \in \mathcal{L} \land P(B) > 0\}$$

the lower and

$$A \subseteq \Omega : (P_{\mathcal{L}})^*(A \mid B) = \sup\{P(A \mid B) \mid P \in \mathcal{L} \land P(B) > 0\}$$

the upper conditional probability of $A$ given $B$.

A class $\mathcal{L}$ of probability measures on $\Omega = \{\omega_1, \ldots, \omega_n\}$ is of type 1, iff there exist functions $R_1$ and $R_2$ from $2^\Omega$ into $[0, 1]$ with:

$$\mathcal{L} = \{P \mid \forall A \subseteq \Omega : R_1(A) \leq P(A) \leq R_2(A)\}$$
Intuition: $P$ is determined by $P(\{\omega_i\})$, $i = 1, \ldots, n$ which corresponds to a point in $\mathbb{R}^n$ with coordinates $(P(\{\omega_1\}), \ldots, P(\{\omega_n\})).$

If $\mathcal{L}$ is type 1, it holds true that:

$$\mathcal{L} \iff \{ (r_1, \ldots, r_n) \in \mathbb{R}^n \mid \exists P : \forall A \subseteq \Omega : (P \mathcal{L})* (A) \leq P(A) \leq (P \mathcal{L})* (A)$$

$$\text{and } r_i = P(\{\omega_i\}), \ i = 1, \ldots, n \}$$
Example

\[ \Omega = \{\omega_1, \omega_2, \omega_3\} \]

\[ \mathcal{L} = \{P \mid \frac{1}{2} \leq P(\{\omega_1, \omega_2\}) \leq 1, \quad \frac{1}{2} \leq P(\{\omega_2, \omega_3\}) \leq 1, \quad \frac{1}{2} \leq P(\{\omega_1, \omega_3\}) \leq 1\} \]

General restriction:

\[ 0 \leq P(\{\omega_i\}) \leq 1 \]

\[ P(\{\omega_1\}) + P(\{\omega_2\}) + P(\{\omega_3\}) = 1 \]

Let \( A_1 = \{\omega_1, \omega_2\}, \ A_2 = \{\omega_2, \omega_3\}, \ A_3 = \{\omega_1, \omega_3\} \)

\[
P_*(A_1) + P_*(A_2) + P_*(A_3) - P_*(A_1 \cap A_2) - P_*(A_2 \cap A_3) - P_*(A_1 \cap A_3) + P_*(A_1 \cap A_2 \cap A_3)
\]

\[
= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - 0 - 0 - 0 + 0 = \frac{3}{2} > 1 = P(A_1 \cup A_2 \cup A_3)
\]
Belief Revision (3)

If $\mathcal{L}$ is type 1 and $(P_{\mathcal{L}})^*(A \cup B) \geq (P_{\mathcal{L}})^*(A) + (P_{\mathcal{L}})^*(B) - (P_{\mathcal{L}})^*(A \cap B)$, then

$$(P_{\mathcal{L}})^*(A \mid B) = \frac{(P_{\mathcal{L}})^*(A \cap B)}{(P_{\mathcal{L}})^*(A \cap B) + (P_{\mathcal{L}})^*(B \cap \overline{A})}$$

and

$$(P_{\mathcal{L}})^*(A \mid B) = \frac{(P_{\mathcal{L}})^*(A \cap B)}{(P_{\mathcal{L}})^*(A \cap B) + (P_{\mathcal{L}})^*(B \cap \overline{A})}$$

Let $\mathcal{L}$ be a class of type 1. $\mathcal{L}$ is of type 2, iff

$$(P_{\mathcal{L}})^*(A_1 \cup \cdots \cup A_n) \geq \sum_{I: \emptyset \neq I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} \cdot (P_{\mathcal{L}})^*(\bigcap_{i \in I} A_i)$$