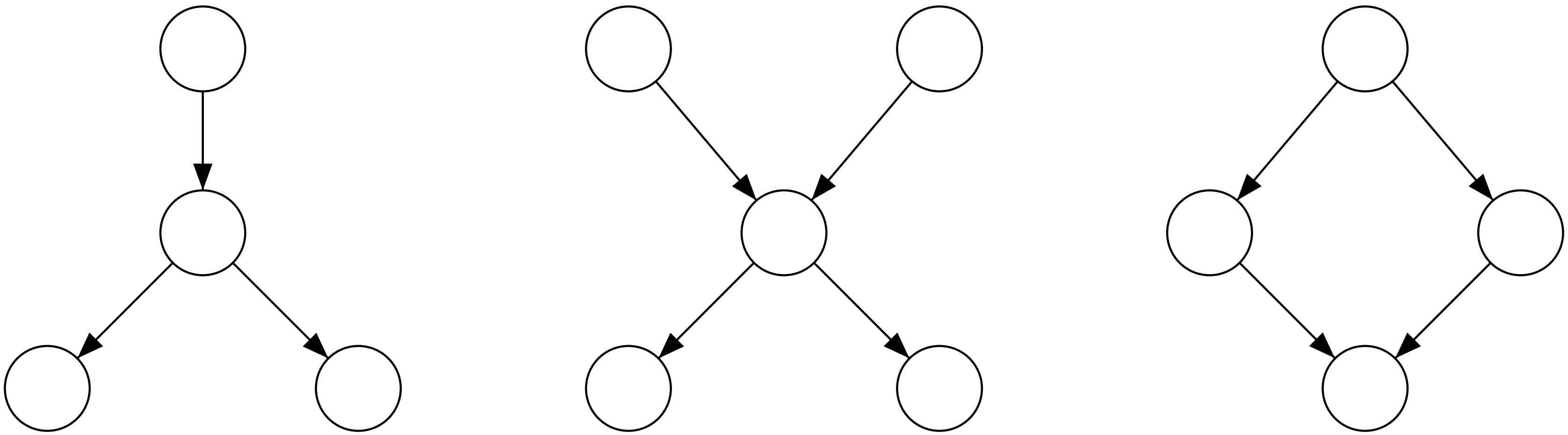
# Inference in Bayes-Networks and Markov-Networks

# Problems



The propagation algorithm as presented can only deal with *trees*.

Can be extended to *polytrees* (i. e. singly connected graphs with multiple parents per node).

However, it cannot handle networks that contain loops!

# Idea

**Main Objectives:**

Transform the cyclic directed graph into a secondary structure without cycles.

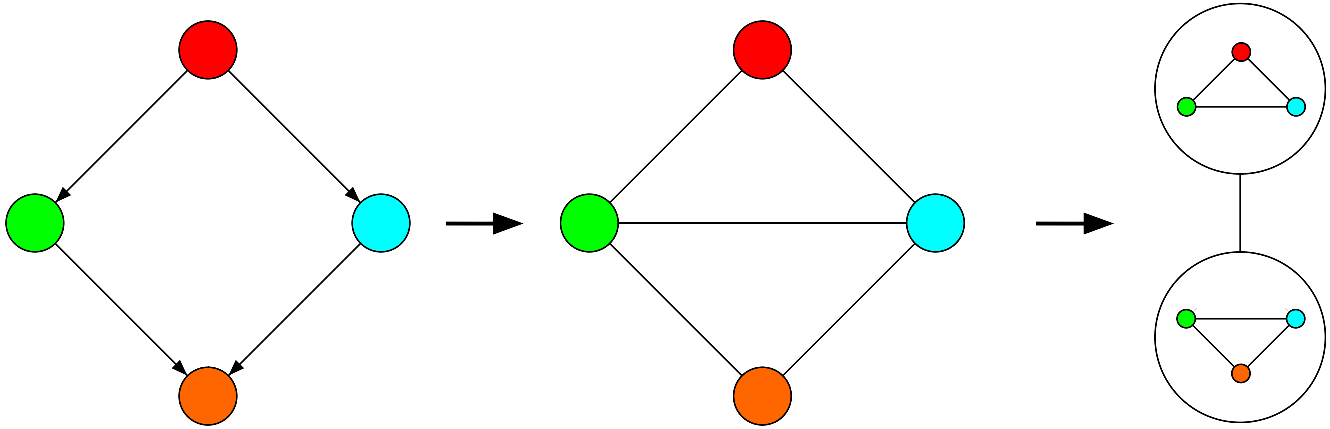
Find a decomposition of the underlying joint distribution.

**Task:**

Combine nodes of the original (primary) graph structure.

These groups form the nodes of a secondary structure.

Find a transformation that yields tree structure.



# Idea (2)

**Secondary Structure:**

We will generate an undirected graph mimicking (some of) the conditional independence statements of the cyclic directed graph.

Maximal cliques are identified and form the nodes of the secondary structure.

Specify a so-called potential function for every clique such that the product of all potentials yields the initial joint distribution.

In order to propagate evidence, create a **tree** from the clique nodes such that the following property is satisfied:

> If two cliques have some attributes in common, then these attributes have to be contained in every clique of the path connecting the two cliques. (called the **running intersection property**, **RIP**)

**Justification:**

Tree: Unique path of evidence propagation.

RIP: Update of an attribute reaches all cliques which contain it.

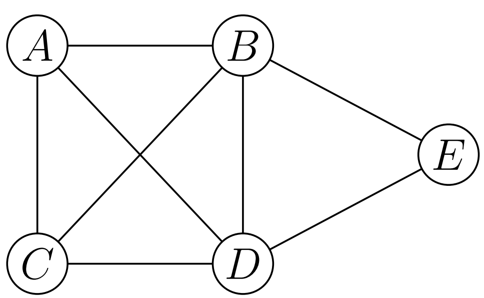
## Complete Graph

An undirected Graph $G = (V, E)$ is called *complete*, if every pair of (distinct) nodes is connected by an edge.

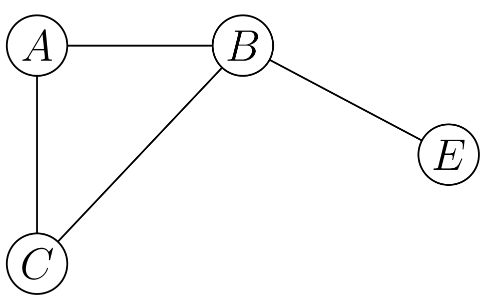
## Induced Subgraph

Let $G = (V, E)$ be an undirected graph and $W \subseteq V$ a selection of nodes. Then, $G_W = (W, E_W)$ is called the *subgraph of* $G$ *induced by* $W$ with $E_W$ being

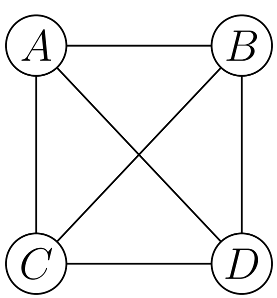
$$E_W = \{(u, v) \in E \mid u, v \in W\}.$$




Incomplete graph




Subgraph $(W, E_W)$
with $W = \{A, B, C, E\}$




Complete (sub)graph

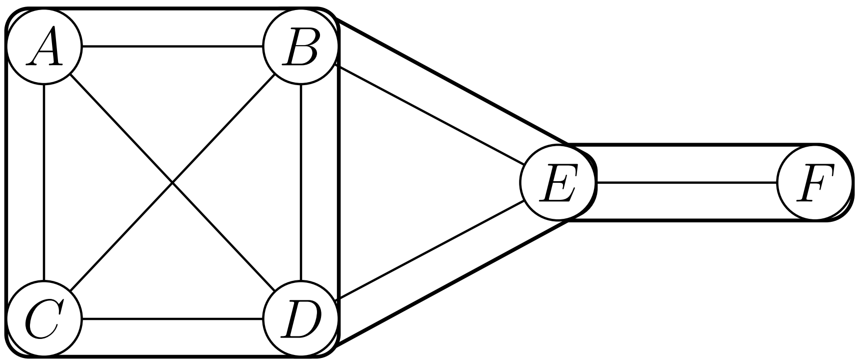
# Prerequisites (2)

## Complete Set, Clique

Let $G = (V, E)$ be an undirected graph. A set $W \subseteq V$ is called *complete* iff it induces a complete subgraph. It is further called a *clique*, iff $W$ is maximal, i. e. it is not possible to add a node to $W$ without violating the completeness condition.

a) $W$ is complete $\Leftrightarrow$ $W$ induces a complete subgraph

b) $W$ is a clique $\Leftrightarrow$ $W$ is complete and maximal




3 cliques

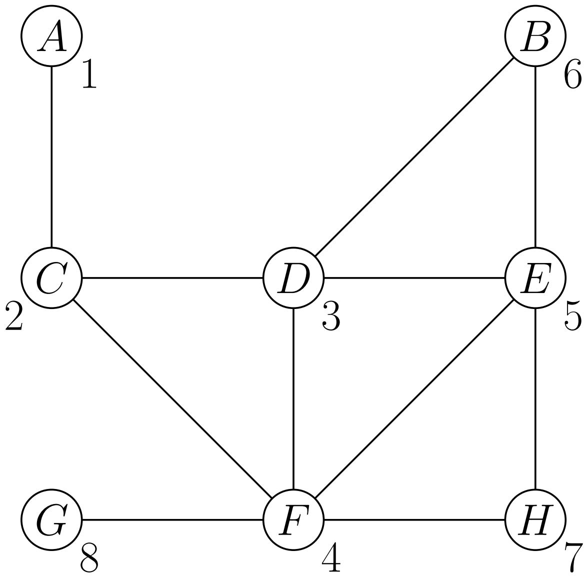
$$C_1 = \{A, B, C, D\}$$
$$C_2 = \{B, D, E\}$$
$$C_3 = \{E, F\}$$

## Perfect Ordering

Let $G = (V, E)$ be an undirected graph with $n$ nodes and $\alpha = \langle v_1, \ldots, v_n \rangle$ a total ordering on $V$. Then, $\alpha$ is called *perfect*, if the following sets

$$\text{adj}(v_i) \cap \{v_1, \ldots, v_{i-1}\} \qquad i = 1, \ldots, n$$

are complete, where $\text{adj}(v_i) = \{w \mid (v_i, w) \in E\}$ returns the adjacent nodes of $v_i$.




$\alpha = \langle A,\ C,\ D,\ F,\ E,\ B,\ H,\ G \rangle$

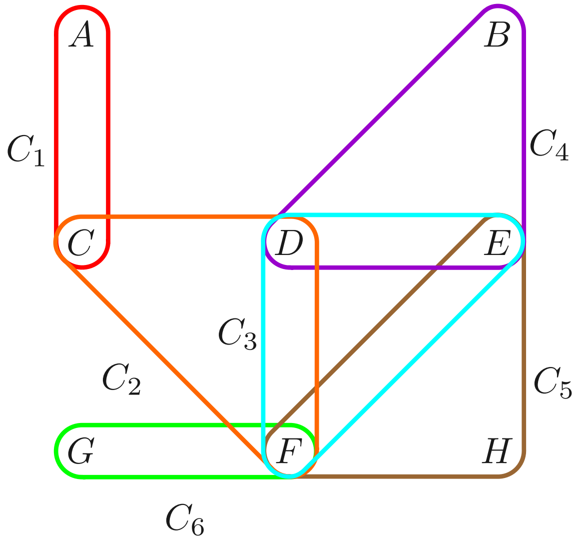
| $i$ | $\text{adj}(v_i)$ | $\text{adj}(v_i) \cap \{v_1, \ldots, v_{i-1}\}$ | | |
|---|---|---|---|---|
| 1 | $\{C\}$ | $\{C\} \cap \emptyset$ | $= \emptyset$ | complete |
| 2 | $\{A, D, F\}$ | $\{A\} \cap \{A, D, F\}$ | $= \{A\}$ | complete |
| 3 | $\{C, B, E, F\}$ | $\{A, C\} \cap \{C, B, E, F\}$ | $= \{C\}$ | complete |
| 4 | $\{G, C, D, E, H\}$ | $\{A, C, D\} \cap \{G, C, D, E, H\}$ | $= \{C, D\}$ | complete |
| 5 | $\{B, D, F, H\}$ | $\{A, C, D, F\} \cap \{B, D, F, H\}$ | $= \{D, F\}$ | complete |
| 6 | $\{D, E\}$ | $\{A, C, D, F, E\} \cap \{D, E\}$ | $= \{D, E\}$ | complete |
| 7 | $\{F, E\}$ | $\{A, C, D, F, E, B\} \cap \{F, E\}$ | $= \{F, E\}$ | complete |
| 8 | $\{F\}$ | $\{A, C, D, F, E, B, H\} \cap \{F\}$ | $= \{F\}$ | complete |

$\alpha$ is a perfect ordering

## Running Intersection Property

Let $G = (V, E)$ be an undirected graph with $p$ cliques. An ordering of these cliques has the *running intersection property (RIP)*, if for every $j > 1$ there exists an $i < j$ such that:

$$C_j \ \cap \ \left(C_1 \cup \cdots \cup C_{j-1}\right) \ \subseteq \ C_i$$

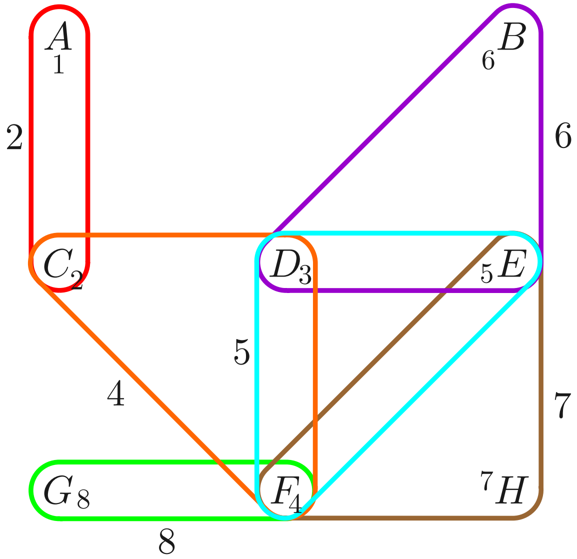



$\xi = \langle C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ C_6 \rangle$

| $j$ | | | | $i$ |
|---|---|---|---|---|
| 2 | $C_2 \cap C_1$ | $= \{C\}$ | $\subseteq C_1$ | 1 |
| 3 | $C_3 \cap (C_1 \cup C_2)$ | $= \{D, F\}$ | $\subseteq C_2$ | 2 |
| 4 | $C_4 \cap (C_1 \cup C_2 \cup C_3)$ | $= \{D, E\}$ | $\subseteq C_3$ | 3 |
| 5 | $C_5 \cap (C_1 \cup C_2 \cup C_3 \cup C_4)$ | $= \{E, F\}$ | $\subseteq C_3$ | 3 |
| 6 | $C_6 \cap (C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5)$ | $= \{F\}$ | $\subseteq C_5$ | 5 |

$\xi$ has running intersection property

If a node ordering $\alpha$ of an undirected graph $G = (V, E)$ is perfect and the cliques of $G$ are ordered according to the highest rank (w. r. t. $\alpha$) of the containing nodes, then this clique ordering has RIP.

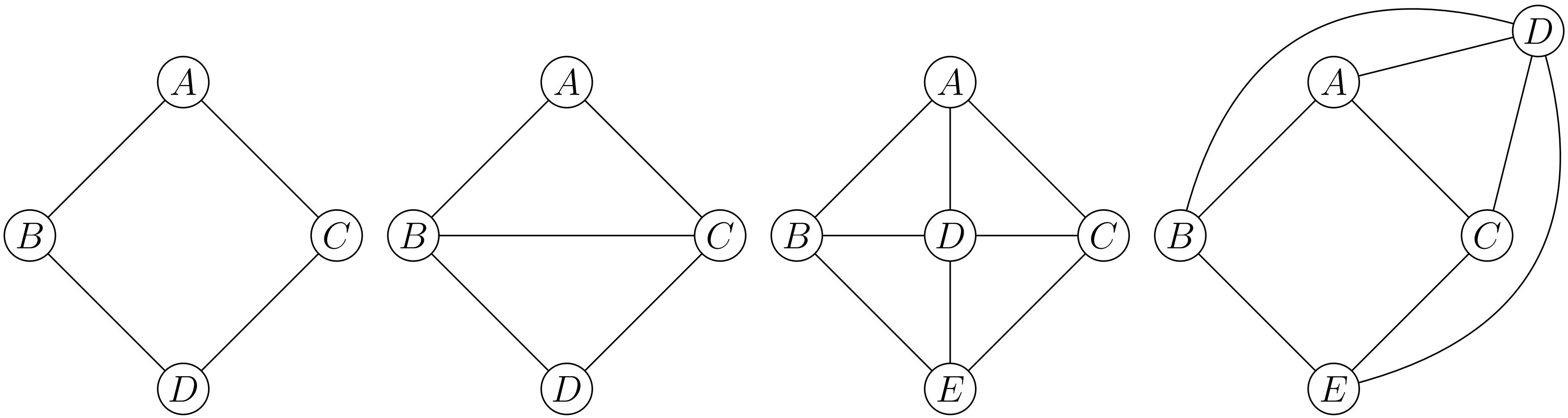



| Clique | Rank | | |
|---|---|---|---|
| $\{A, C\}$ | $\max\{\alpha(A), \alpha(C)\}$ | $= 2$ | $\rightarrow C_1$ |
| $\{C, D, F\}$ | $\max\{\alpha(C), \alpha(D), \alpha(F)\}$ | $= 4$ | $\rightarrow C_2$ |
| $\{D, E, F\}$ | $\max\{\alpha(D), \alpha(E), \alpha(F)\}$ | $= 5$ | $\rightarrow C_3$ |
| $\{B, D, E\}$ | $\max\{\alpha(B), \alpha(D), \alpha(E)\}$ | $= 6$ | $\rightarrow C_4$ |
| $\{F, E, H\}$ | $\max\{\alpha(F), \alpha(E), \alpha(H)\}$ | $= 7$ | $\rightarrow C_5$ |
| $\{F, G\}$ | $\max\{\alpha(F), \alpha(G)\}$ | $= 8$ | $\rightarrow C_6$ |

How to get a perfect ordering?

## Triangulated Graph

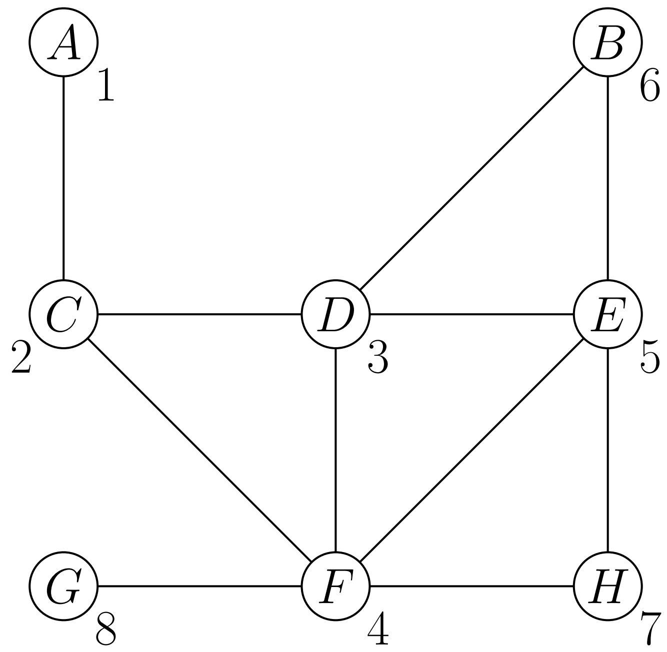
An undirected graph is called *triangulated* if every simple loop (i. e. path with identical start and end node but with any other node occurring at most once) of length greater 3 has a chord.




not triangulated | triangulated | not triangulated | no chord for $\langle A, B, E, C \rangle$

## Maximum Cardinality Search

Let $G = (V, E)$ be an undirected graph. An ordering according *maximum cardinality search (MCS)* is obtained by first assigning 1 to an arbitray node. If $n$ numbers are assigned the node that is connected to most of the nodes already numbered gets assigned number $n + 1$.




3 can be assigned to $D$ or $F$

6 can be assigned to $H$ or $B$

# Triangulated Graphs (3)

An undirected graph is triangulated iff the ordering obtained by MCS is perfect.

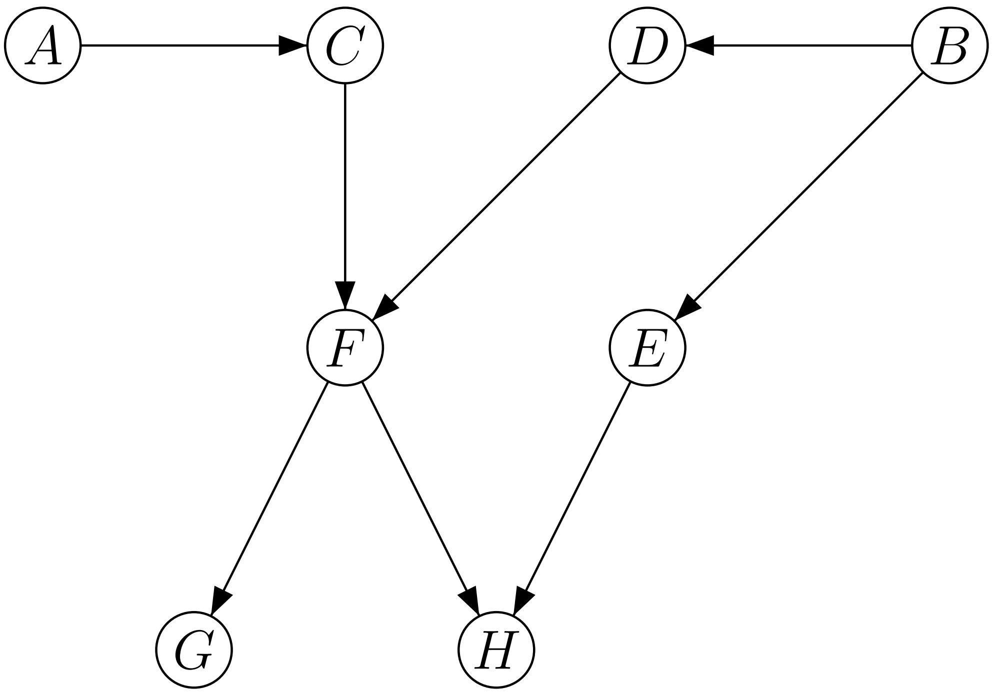
To check whether a graph is triangulated is efficient to implement. The optimization problem that is related to the triangulation task is NP-hard. However, there are good heuristics.

**Moral Graph** (Repetition)

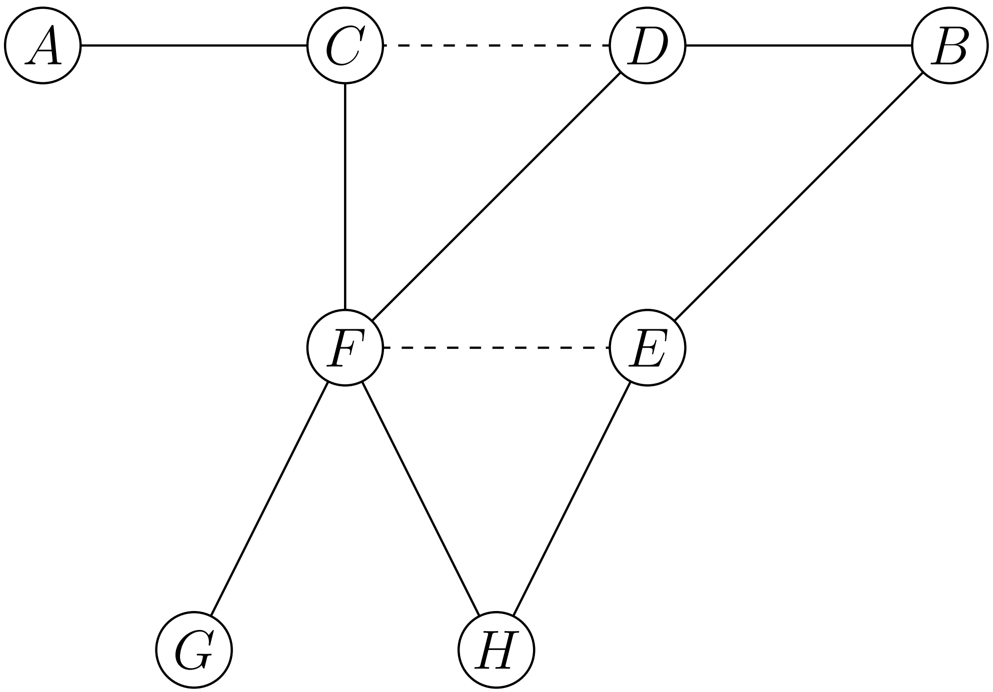
Let $G = (V, E)$ be a directed acyclic graph. If $u, w \in W$ are parents of $v \in V$ connect $u$ and $w$ with an (arbitrarily oriented) edge. After the removal of all edge directions the resulting graph $G_m = (V, E')$ is called the *moral graph* of $G$.

# Join-Tree Construction (1)

Given directed graph.

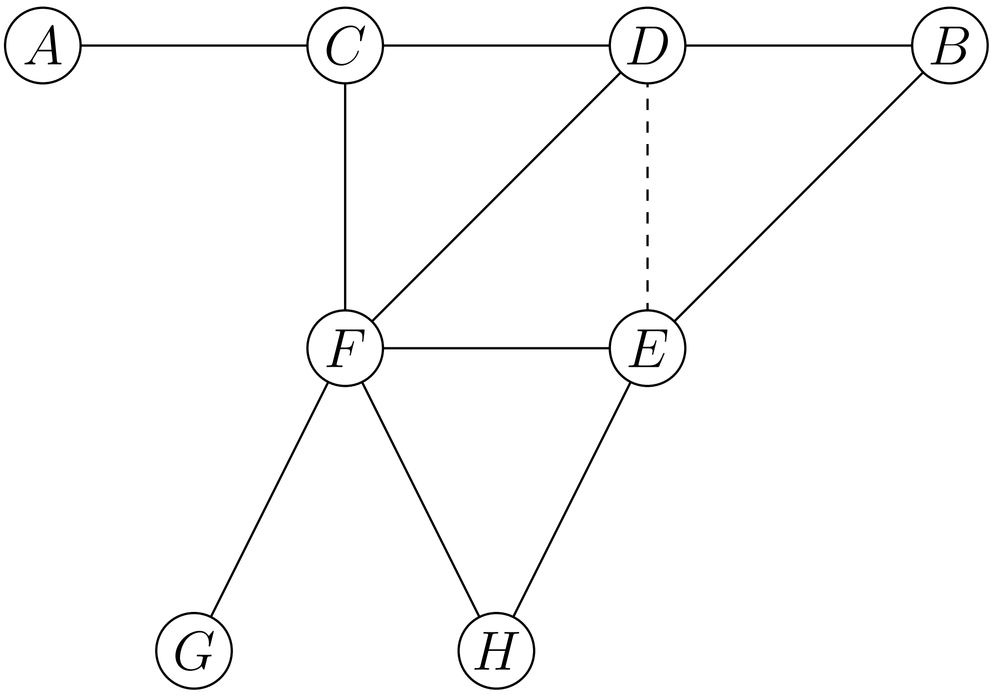


# Join-Tree Construction (2)



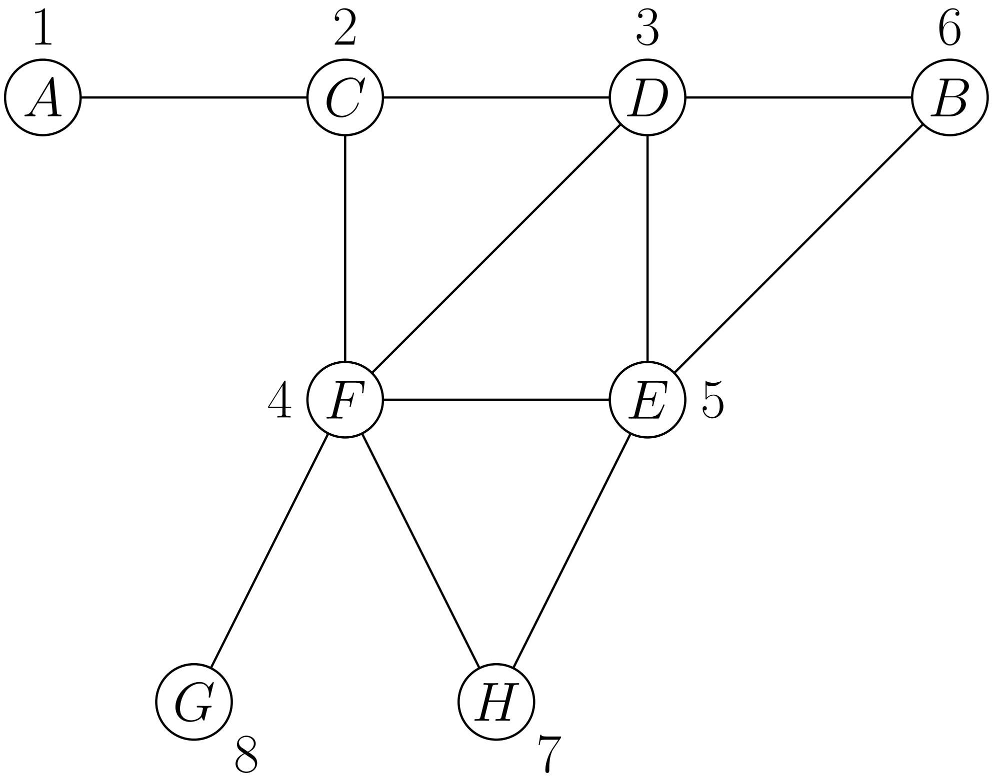

- Moral graph

# Join-Tree Construction (3)



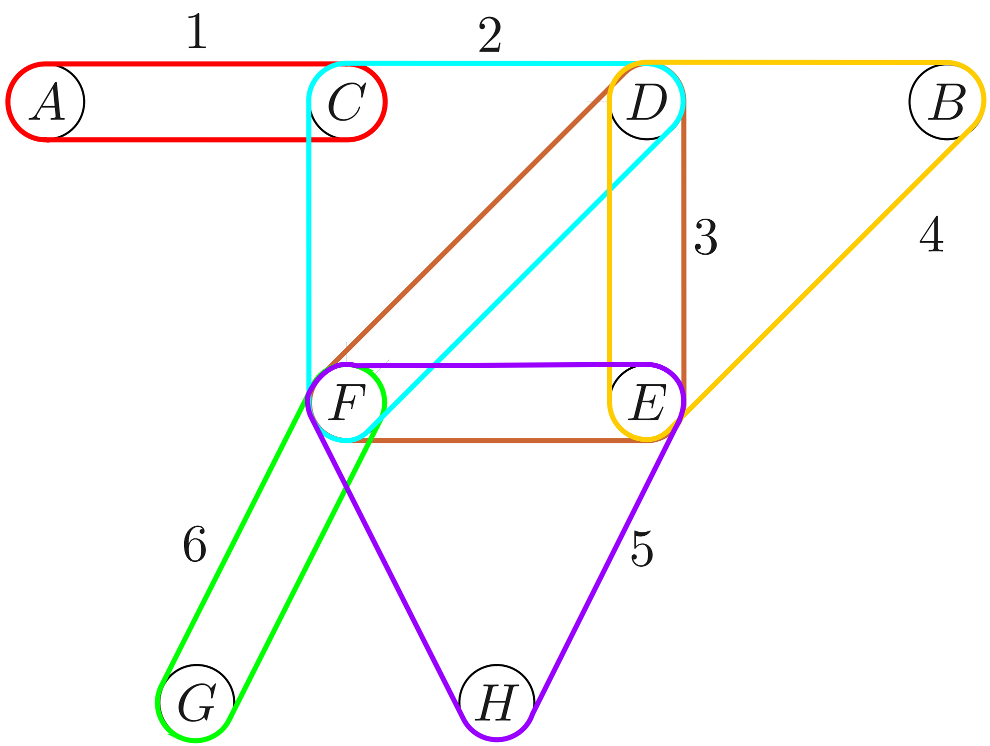

- Moral graph
- Triangulated graph

# Join-Tree Construction (4)



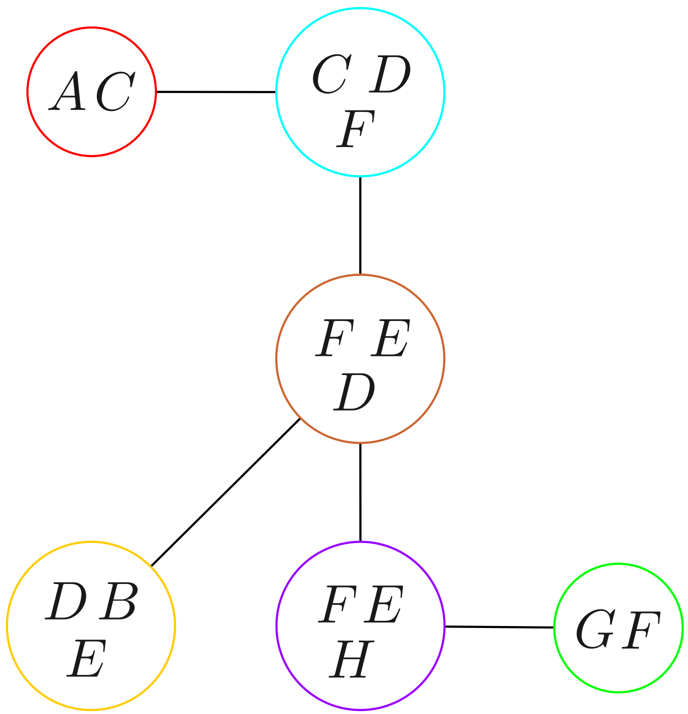

- Moral graph
- Triangulated graph
- MCS yields perfect ordering

# Join-Tree Construction (5)




- Moral graph
- Triangulated graph
- MCS yields perfect ordering
- Clique order has RIP

# Join-Tree Construction (6)




- Moral graph
- Triangulated graph
- MCS yields perfect ordering
- Clique order has RIP
- Form a join-tree

  Two cliques can be connected if they have a non-empty intersection. The generation of the tree follows the RIP. In case of a tie, connect cliques with the largest intersection. (e. g. $DBE$—$FED$ instead of $DBE$—$CFD$) Break remaining ties arbitrarily.

# Example: Expert Knowledge

**Qualitative knowledge:**

Metastatic cancer is a possible cause of brain tumor, and is also an explanation for increased total serum calcium. In turn, either of these could explain a patient falling into a coma. Severe headache is also possibly associated with a brain tumor.

**Special case:**

The patient has heavy headache.

**Query:**

Will the patient fall into coma?

# Example: Choice of State Space

| Attribute | | Possible Values | |
|---|---|---|---|
| $A$ | metastatic cancer | $\text{dom}(A) = \{a_1, a_2\}$ | $\cdot_1 = \text{existing}$ |
| $B$ | increased total serum calcium | $\text{dom}(B) = \{b_1, b_2\}$ | $\cdot_2 = \text{notexisting}$ |
| $C$ | brain tumor | $\text{dom}(C) = \{c_1, c_2\}$ | |
| $D$ | coma | $\text{dom}(D) = \{d_1, d_2\}$ | |
| $E$ | severe headache | $\text{dom}(E) = \{e_1, e_2\}$ | |

Exhaustive state space:

$$\Omega = \text{dom}(A) \times \text{dom}(B) \times \text{dom}(C) \times \text{dom}(D) \times \text{dom}(E)$$

Marginal and conditional probabilities have to be specified!

# Example: Qualitative Knowledge

$$\left.\begin{aligned} P(e_1 \mid c_1) &= 0.8 \\ P(e_1 \mid c_2) &= 0.6 \end{aligned}\right\}$$ headaches common, but more common if tumor present

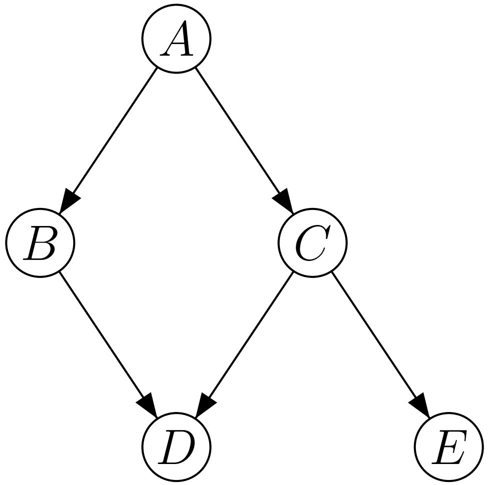
$$\left.\begin{aligned} P(d_1 \mid b_1, c_1) &= 0.8 \\ P(d_1 \mid b_1, c_2) &= 0.8 \\ P(d_1 \mid b_2, c_1) &= 0.8 \\ P(d_1 \mid b_2, c_2) &= 0.05 \end{aligned}\right\}$$ coma rare but common, if either cause is present

$$\left.\begin{aligned} P(b_1 \mid a_1) &= 0.8 \\ P(b_1 \mid a_2) &= 0.2 \end{aligned}\right\}$$ increased calcium uncommon, but common consequence of metastases

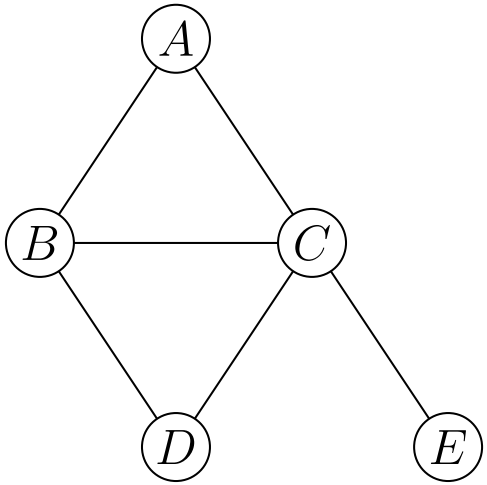
$$\left.\begin{aligned} P(c_1 \mid a_1) &= 0.2 \\ P(c_1 \mid a_2) &= 0.05 \end{aligned}\right\}$$ brain tumor rare, and uncommon consequence of metastases

$$\left. P(a_1) = 0.2 \right\}$$ incidence of metastatic cancer in relevant clinic

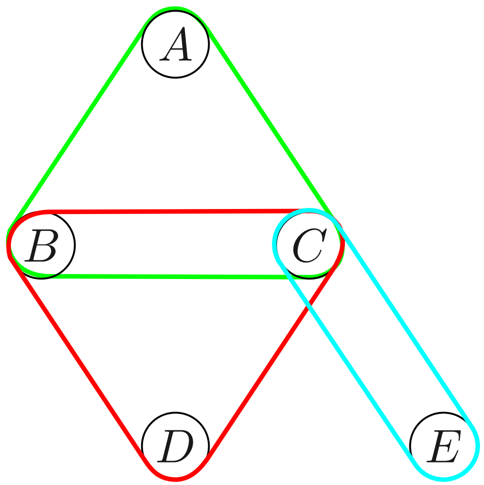
Example: Metastatic Cancer




Dependencies




Moralization/Triangulation




MCS, hyper graph


Clique tree with separator sets

# Propagation on Cliques (3)

Quantitative knowledge:

| $(a, b, c)$ | $P(a, b, c)$ |
|---|---|
| $a_1, b_1, c_1$ | 0.032 |
| $a_2, b_1, c_1$ | 0.008 |
| $\vdots$ | $\vdots$ |
| $a_2, b_2, c_2$ | 0.608 |

| $(b, c, d)$ | $P(b, c, d)$ |
|---|---|
| $b_1, c_1, d_1$ | 0.032 |
| $b_2, c_1, d_1$ | 0.032 |
| $\vdots$ | $\vdots$ |
| $b_2, c_2, d_2$ | 0.608 |

| $(c, e)$ | $P(c, e)$ |
|---|---|
| $c_1, e_1$ | 0.064 |
| $c_2, e_1$ | 0.552 |
| $c_1, e_2$ | 0.016 |
| $c_2, e_2$ | 0.368 |

Potential representation:

$$\begin{aligned} P(A, B, C, D, E, ) &= P(A \mid \emptyset)P(B \mid A)P(C \mid A)P(D \mid BC)P(E \mid C) \\ &= \frac{P(A, B, C)P(B, C, D), P(C, E)}{P(BC)P(C)} \end{aligned}$$

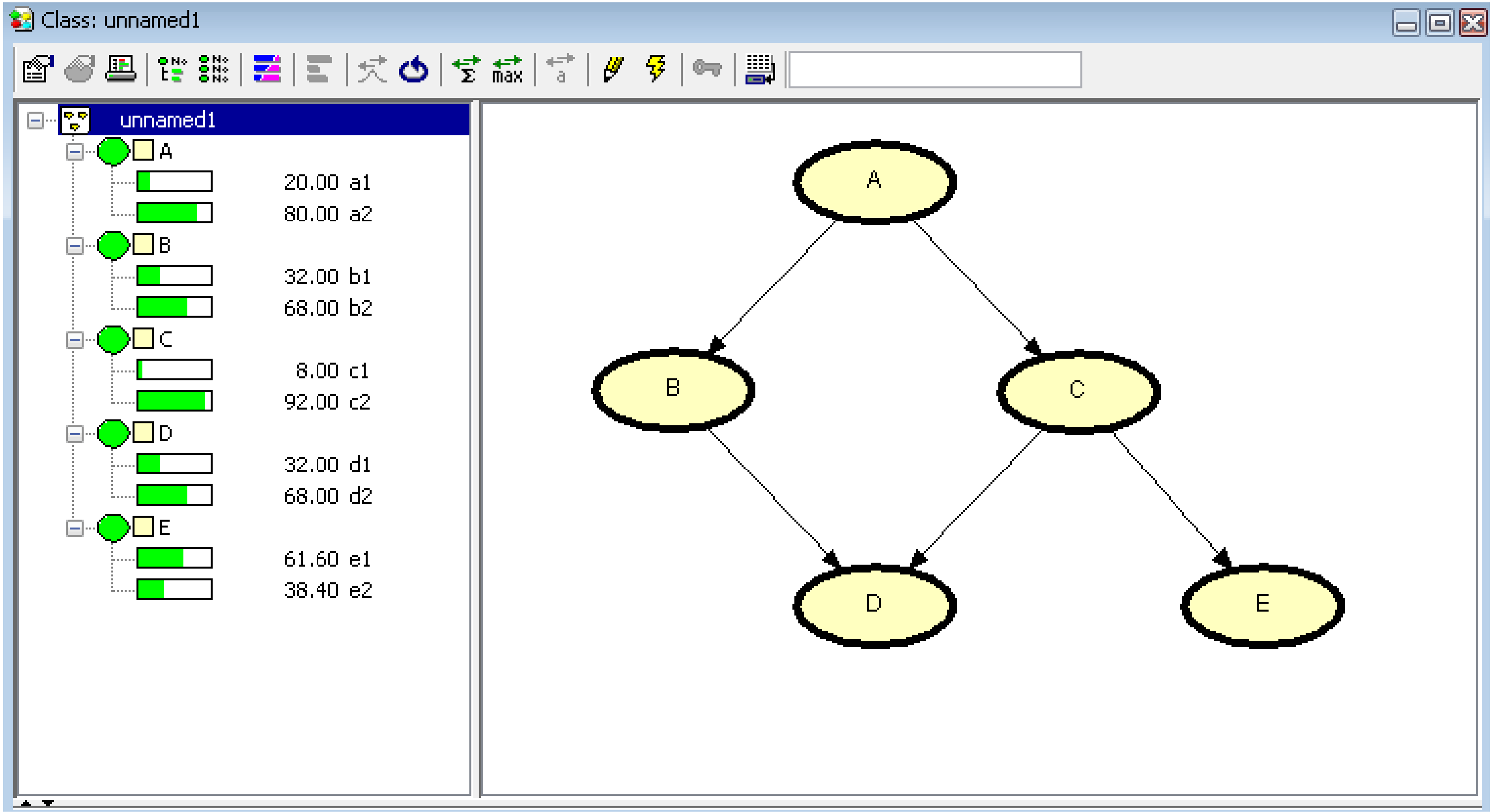
# Propagation on Cliques (4)

Propagation:

$$P(d_1) = 0.32, \quad \text{evidence } E = e_1, \quad \text{desired:} \quad P^*(\ldots) = P(\cdot \mid \{e_1\})$$

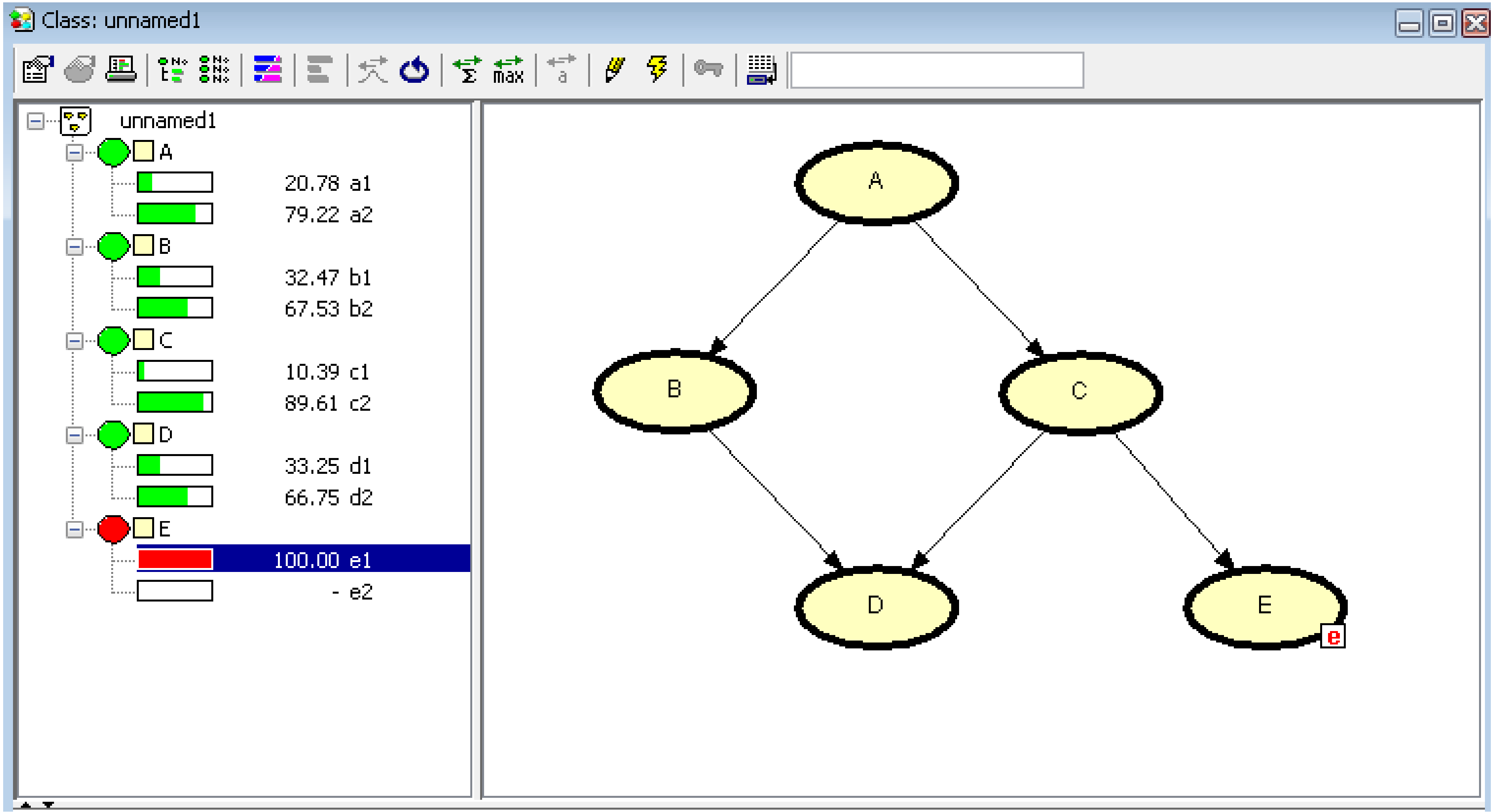
| | | |
|---|---|---|
| $P^*(c)$ | $= P(c \mid e_1)$ | conditional marginal distribution |
| $P^*(b,c,d)$ | $= \dfrac{P(b,c,d)}{P(c)} P(c \mid e_1)$ | multipl./division with separation prob. |
| $P(b,c,d),\ P^*(b,c)$ | | calculate marginal distributions |
| $P^*(a,b,c)$ | $= \dfrac{P(a,b,c)}{P(b,c)} P(b,c \mid e_1)$ | multipl./division with separation prob. |
| $P^*(d_1)$ | $= P(d_1 \mid e_1) = 0.33$ | |

# Propagation on Cliques (5)




Marginal distributions in the HUGIN tool.

# Propagation on Cliques (6)




Conditional marginal distributions with evidence $E = e_1$

**Potential Representation**

Let $V = \{X_j\}$ be a set of random variables $X_j : \Omega \to \mathrm{dom}(X_j)$ and $P$ the joint distribution over $V$. Further, let

$$\{W_i \mid W_i \subseteq V, 1 \leq i \leq p\}$$

a family of subsets of $V$ with associated functions

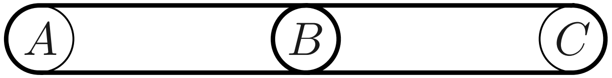
$$\psi_i : \bigtimes_{X_j \in W_i} \mathrm{dom}(X_j) \to \mathbb{R}$$

It is said that $P(V)$ *factorizes* according $\big(\{W_1, \ldots, W_p\}, \{\psi_1, \ldots, \psi_p\}\big)$ if $P(V)$ can be written as:

$$P(v) = k \cdot \prod_{i=1}^{p} \psi_i(w_i)$$

where $k \in \mathbb{R}$, $w_i$ is a realization of $W_i$ that meets the values of $v$.

# Example




$V = \{A, B, C\},\ W_1 = \{A, B\},\ W_2 = \{B, C\}$

$\text{dom}(A) = \{a_1, a_2\}$
$\text{dom}(B) = \{b_1, b_2\}$
$\text{dom}(C) = \{c_1, c_2\}$

$P(a, b, c) = \frac{1}{8}$

$\psi_1 : \{a_1, a_2\} \times \{b_1, b_2\} \to \mathbb{R}$
$\psi_2 : \{b_1, b_2\} \times \{c_1, c_2\} \to \mathbb{R}$

$\psi_1(a, b) = \frac{1}{4}$
$\psi_2(b, c) = \frac{1}{2}$

$(\{W_1, W_2\}, \{\psi_1, \psi_2\})$ is a potential representation of $P$.

# Factorization of a Belief Network

Let $(V, E, P)$ be an belief network and $\{C_1, \ldots, C_p\}$ the cliques of the join tree. For every node $v \in V$ choose a clique $C$ such that $v$ and all of its parents are contained in $C$, i. e. $\{v\} \cup c(v) \subseteq C$. The chosen clique is designated as $f(v)$.

To arrive at a factorization $(\{C_1, \ldots, C_p\}, \{\psi_1, \ldots, \psi_p\})$ of $P$ the factor potentials are:

$$\psi_i(c_i) = \prod_{v:f(v)=C_i} P(v \mid c(v))$$

**Separator Sets and Residual Sets**

Let $\{C_1, \ldots, C_p\}$ be a set of cliques w. r. t. $V$. The sets

$$S_i = C_i \cap (C_1 \cup \cdots \cup C_{i-1}), \qquad i = 1, \ldots, p, \qquad S_1 = \emptyset$$

are called *separator sets* with their corresponding *residual sets*

$$R_i = C_i \backslash S_i$$

# Decomposition w. r. t. a Join-Tree

Given a clique ordering $\{C_1, \ldots, C_p\}$ that satisfies the RIP, we can easily conclude the following separation statements:

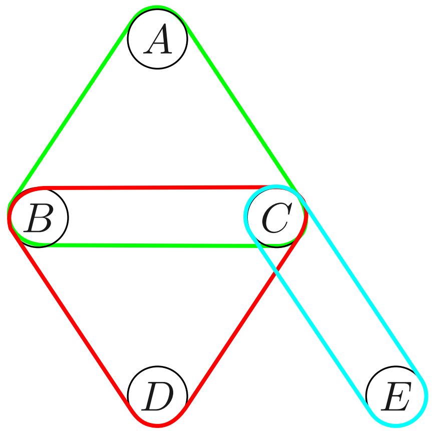
$$R_i \perp\!\!\!\perp (C_1 \cup \cdots \cup C_{i-1}) \backslash S_i \mid S_i \qquad \text{for } i > 1$$

Hence, we can formulate the following factorization:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{p} P(R_i \mid S_i),$$

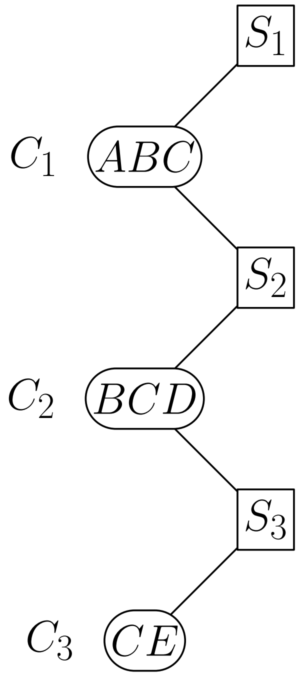
which also gives us a representation in terms of conditional probabilities (as for directed graphs before).

# Example




$$S_1 = \emptyset \qquad R_1 = \{A, B, C\}$$
$$S_2 = \{B, C\} \qquad R_2 = \{D\}$$
$$S_3 = \{C\} \qquad R_3 = \{E\}$$

$$f(A) = C_1$$
$$f(B) = C_1$$
$$f(C) = C_1$$
$$f(D) = C_2$$
$$f(E) = C_3$$




$$\psi_1(C_1) = P(A, B, C \mid \emptyset) = P(A) \cdot P(C \mid A) \cdot P(B|A)$$
$$\psi_2(C_2) = P(D \mid B, C)$$
$$\psi_3(C_3) = P(E \mid C)$$

Propagation is accomplished by sending messages across the cliques in the tree. The emerging potentials are maintained by each clique.

## Main Idea

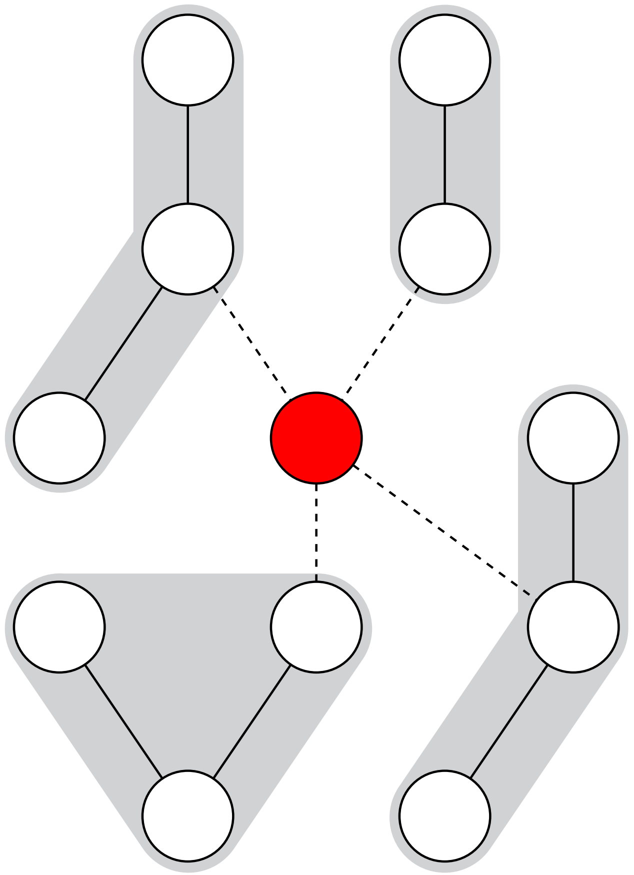
Incorporate evidence into the clique potentials.

Since we are dealing with a tree structure, exploit the fact that a clique "separates" all its neighboring cliques (and their respective subtrees) from each other.

Apply a message passing scheme to inform neighboring cliques about evidence.

Since we do not have edge directions, we will only need one type of message.

After having updated all cliques' potentials, we marginalize (and normalize) to get the probabilities of single attributes.



# Incorporating Evidence

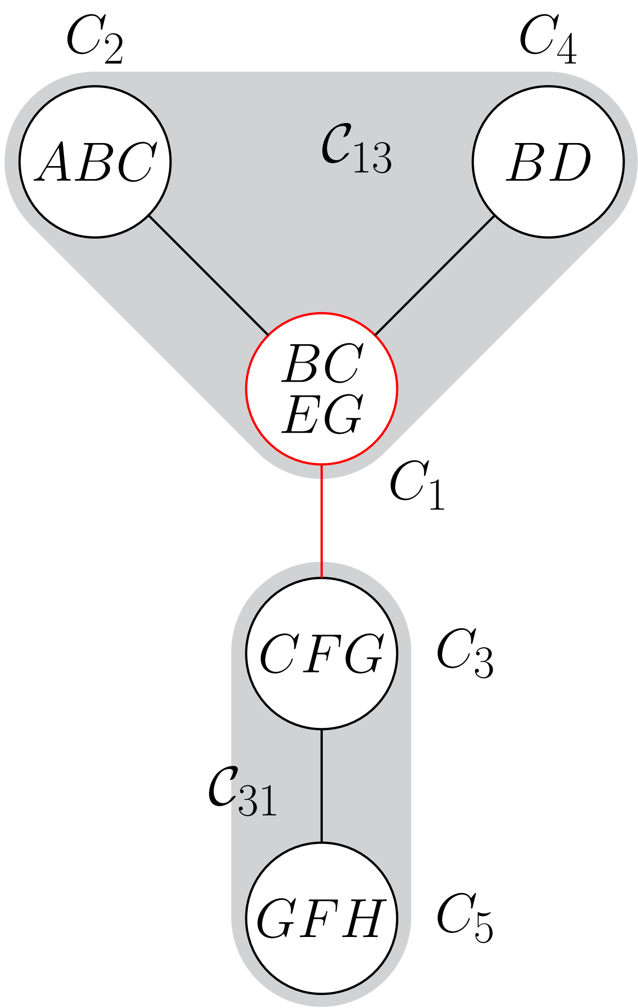
Every clique $C_i$ maintains a potential function $\psi_i$.

If for an attribute $E$ some evidence $e$ becomes known, we alter all potential functions of cliques containing $E$ as follows:

$$\psi_i^*(c_i) = \begin{cases} 0, & \text{if a value in } c_i \text{ is inconsistent with } e \\ \psi_i(c_i), & \text{otherwise} \end{cases}$$

All other potential functions are unchanged.

# Notation and Nomenclature




In general:

Clique $C_i$ has $q$ neighboring cliques $B_1, \ldots, B_q$.

$\mathcal{C}_{ij}$ is the set of cliques in the subtree containing $C_i$ after dropping the link to $B_j$.

$X_{ij}$ is the set of attributes in the cliques of $\mathcal{C}_{ij}$.

$V = X_{ij} \cup X_{ji}$ (complementary sets)

$S_{ij} = S_{ji} = C_i \cap C_j$ (not shown here)

$R_{ij} = X_{ij} \setminus S_{ij}$ (not shown here)

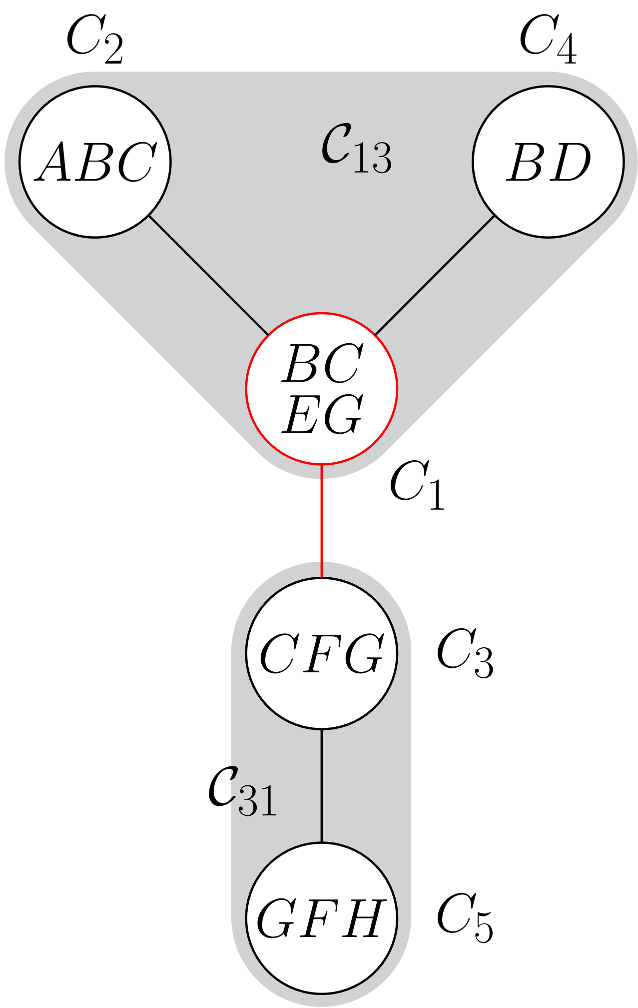
Here:

Neighbors of $C_1$: $\{C_2, C_4, C_3\}$, $\mathcal{C}_{13} = \{C_1, C_2, C_4\}$

$X_{13} = \{A, B, C, D, E, G\}$, $S_{13} = \{C, G\}$

$V = X_{13} \cup X_{31} = \{A, B, C, D, E, F, G, H\}$

$R_{13} = \{A, B, D, E\}$, $R_{31} = \{F, H\}$

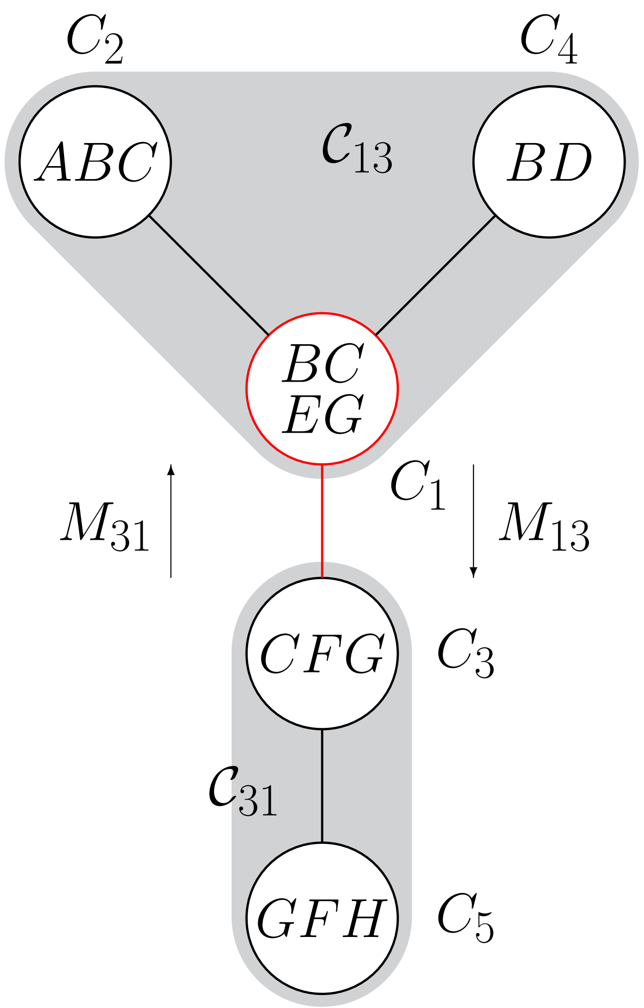



**Task:** Calculate $P(s_{ij})$:

$$\begin{aligned} V \setminus S_{ij} &= (X_{ij} \cup X_{ji}) \setminus S_{ij} \\ &= (X_{ij} \setminus S_{ij}) \ \cup \ (X_{ji} \setminus S_{ij}) \\ &= R_{ij} \ \cup \ R_{ji} \end{aligned}$$

$$\begin{aligned} V \setminus S_{13} &= (X_{13} \cup X_{31}) \setminus S_{13} \\ &= R_{13} \ \cup \ R_{31} \\ V \setminus \{C, G\} &= \{A, B, D, E\} \ \cup \ \{F, H\} \\ &= \{A, B, D, E, F, H\} \end{aligned}$$

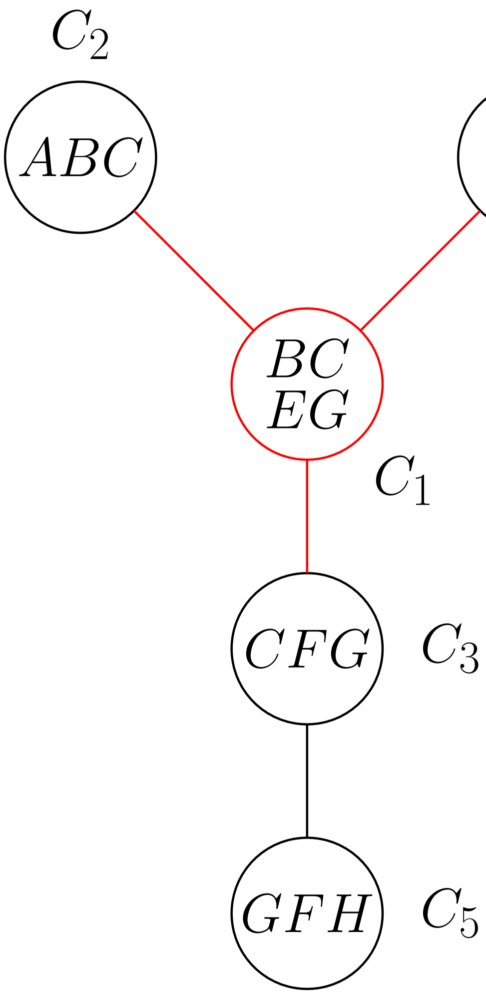
Note: $R_{ij}$ is the set of attributes that are in $C_i$'s subtree but not in $B_j$'s. Therefore, $R_{ij}$ and $R_{ji}$ are always **disjoint**.




**Task:** Calculate $P(s_{ij})$:

$$\begin{aligned}
P(s_{ij}) &= \sum_{v \setminus s_{ij}} \prod_{k=1}^{m} \psi_k(c_k) \\
&\overset{\text{last slide}}{=} \sum_{r_{ij} \cup r_{ji}} \prod_{k=1}^{m} \psi_k(c_k) \\
&\overset{\text{sum rule}}{=} \left( \sum_{r_{ij}} \prod_{c_k \in \mathcal{C}_{ij}} \psi_k(c_k) \right) \cdot \left( \sum_{r_{ji}} \prod_{c_k \in \mathcal{C}_{ji}} \psi_k(c_k) \right) \\
&= M_{ij}(s_{ij}) \cdot M_{ji}(s_{ij})
\end{aligned}$$

$M_{ij}$ is the message sent from $C_i$ to neighbor $B_j$ and vice versa.

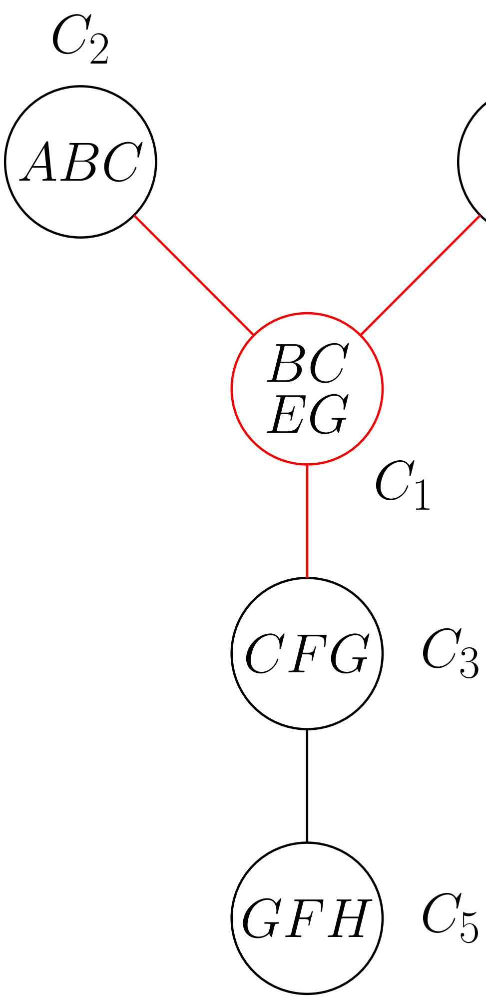



**Task:** Calculate $P(c_i)$:

$$
\begin{aligned}
V \setminus C_i &= \left( \bigcup_{k=1}^{q} X_{ki} \right) \setminus C_i \\
&= \bigcup_{k=1}^{q} \left( X_{ki} \setminus C_i \right) \\
&= \bigcup_{k=1}^{q} R_{ki}
\end{aligned}
$$

Example:

$$
\begin{aligned}
V \setminus C_1 &= R_{21} \cup R_{41} \cup R_{31} \\
\{A, D, F, H\} &= \{A\} \cup \{D\} \cup \{F, H\}
\end{aligned}
$$




**Task:** Calculate $P(c_i)$:

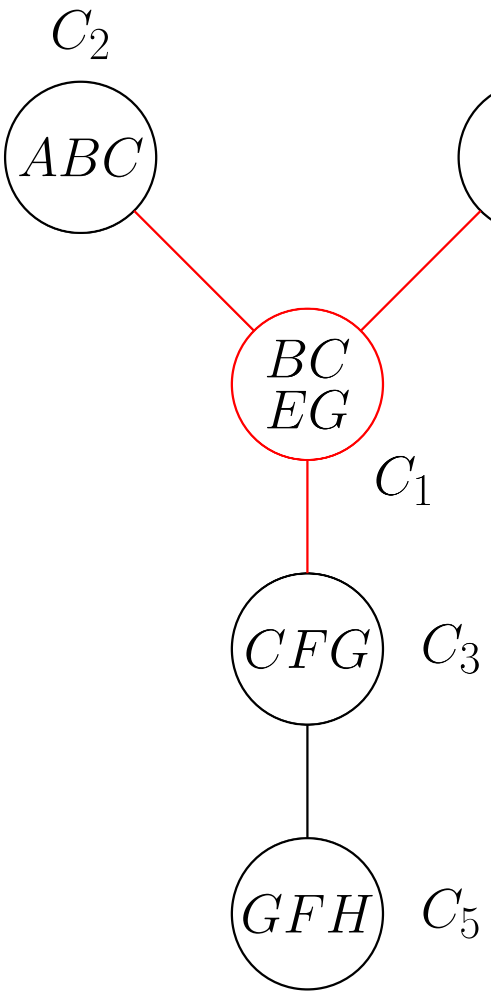
$$P(c_i) = \underbrace{\sum_{v \setminus c_i}}_{\text{Marginalization}} \underbrace{\prod_{j=1}^{m} \psi_j(c_j)}_{\text{Decomposition}}$$

$$= \psi_i(c_i) \sum_{v \setminus c_i} \prod_{i \neq j} \psi_j(c_j)$$

$$= \psi_i(c_i) \sum_{r_{1i} \cup \cdots \cup r_{qi}} \prod_{i \neq j} \psi_j(c_j)$$

$$= \psi_i(c_i) \underbrace{\left( \sum_{r_{1i}} \prod_{c_k \in \mathcal{C}_{1i}} \psi_k(c_k) \right)}_{M_{1i}(s_{ij})} \cdots \underbrace{\left( \sum_{r_{qi}} \prod_{c_k \in \mathcal{C}_{qi}} \psi_k(c_k) \right)}_{M_{qi}(s_{ij})}$$

$$= \psi_i(c_i) \prod_{j=1}^{q} M_{ji}(s_{ij})$$




Example: $P(c_1)$:

$$P(c_1) \;=\; \psi_1(c_1) M_{21}(s_{12}) M_{41}(s_{14}) M_{31}(s_{13})$$

$M_{ij}(s_{ij})$ can be simplified further (without proof):

$$\begin{aligned} M_{ij}(s_{ij}) \;&=\; \sum_{r_{ij}} \prod_{c_k \in \mathcal{C}_{ij}} \psi_k(c_k) \\ &=\; \sum_{c_i \setminus s_{ij}} \psi_i(c_i) \prod_{k \neq j} M_{ki}(s_{ki}) \end{aligned}$$

# Final Algorithm

**Input:** Join tree $(\mathcal{C}, \Psi)$ over set of variables $V$ and evidence $E = e$.

**Output:** The a-posteriori probability $P(x_i \,|\, e)$ for every non-evidential $X_i$.

**Initialization:** Incorporate evidence $E = e$ into potential functions.

**Iterations:**

1. For every clique $C_i$ do: For every neighbor $B_j$ of $C_i$ do: If $C_i$ has received all messages from the *other* neighbors, calculate and send $M_{ij}(s_{ij})$ to $B_j$.
2. Repeat step 1 until no message is calculated.
3. Calculate the joint probability distribution for every clique:

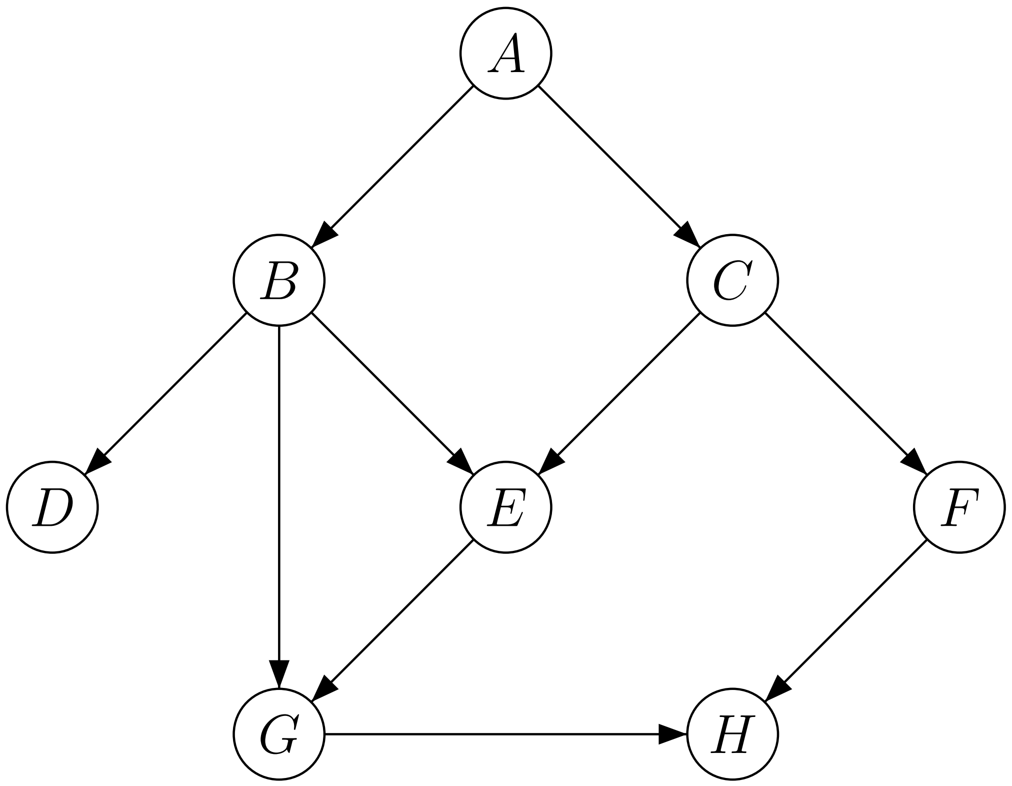
$$P(c_i) \;\propto\; \psi_i(c_i) \prod_{j=1}^{q} M_{ji}(s_{ij})$$

4. For every $X \in V$ calculate the a-posteriori probability:

$$P(x_i \mid e) \;=\; \sum_{c_k \setminus x_i} P(c_k)$$

where $C_k$ is the smallest clique containing $X_i$.

# Example: Putting it together

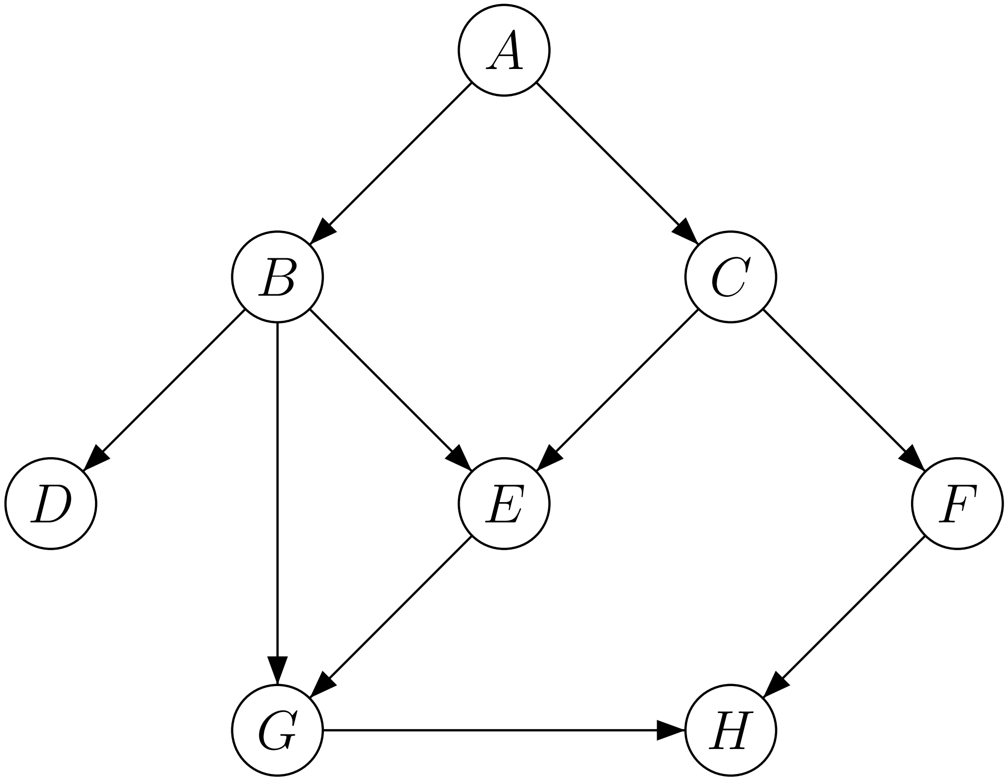



**Goals:** Find the marginal distributions and update them when evidence $H = h_1$ becomes known.

**Steps:**

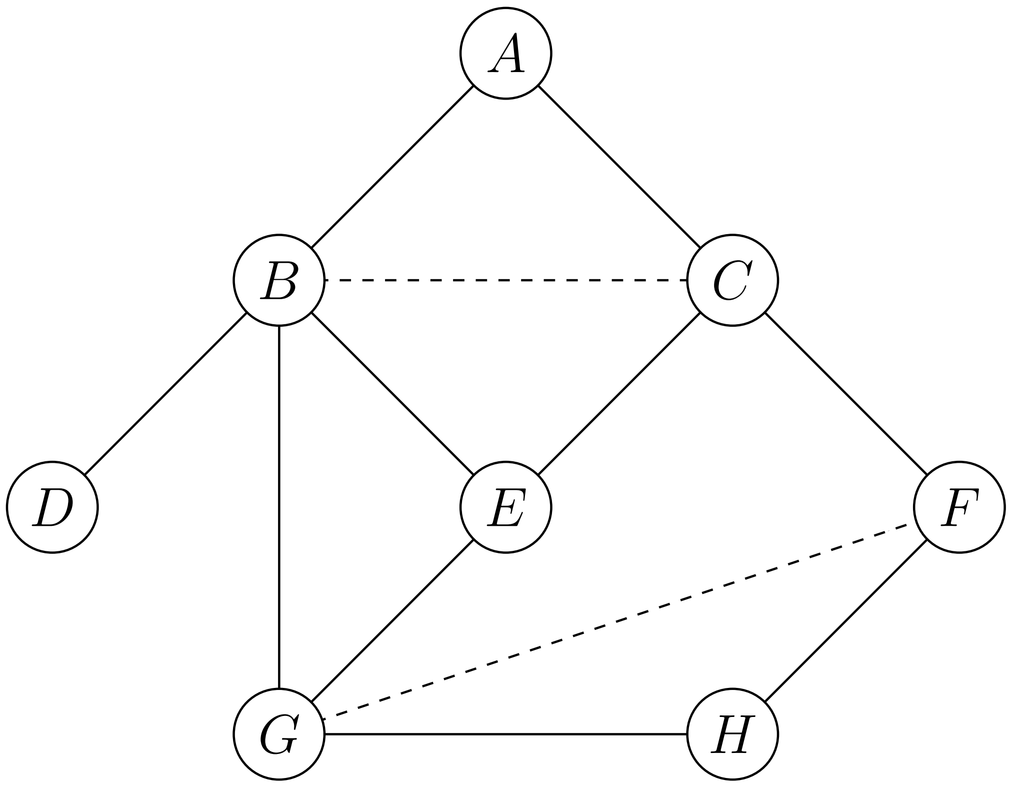
1. Transform network into join-tree.
2. Specify factor potentials.
3. Propagate "zero" evidence to obtain the marginals before evidence is present.
4. Update factor potentials w. r. t. the evidence and do another propagation run.

# Example: Step 1: Find a Join-Tree




**Join-Tree creation:**

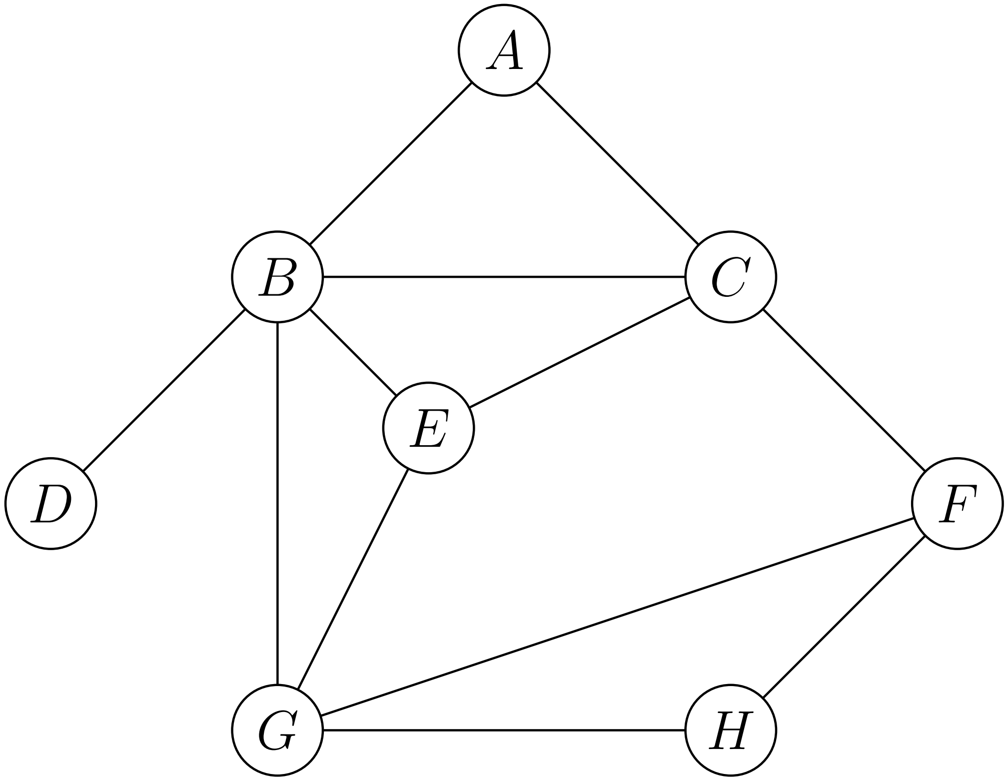
# Example: Step 1: Find a Join-Tree




**Join-Tree creation:**

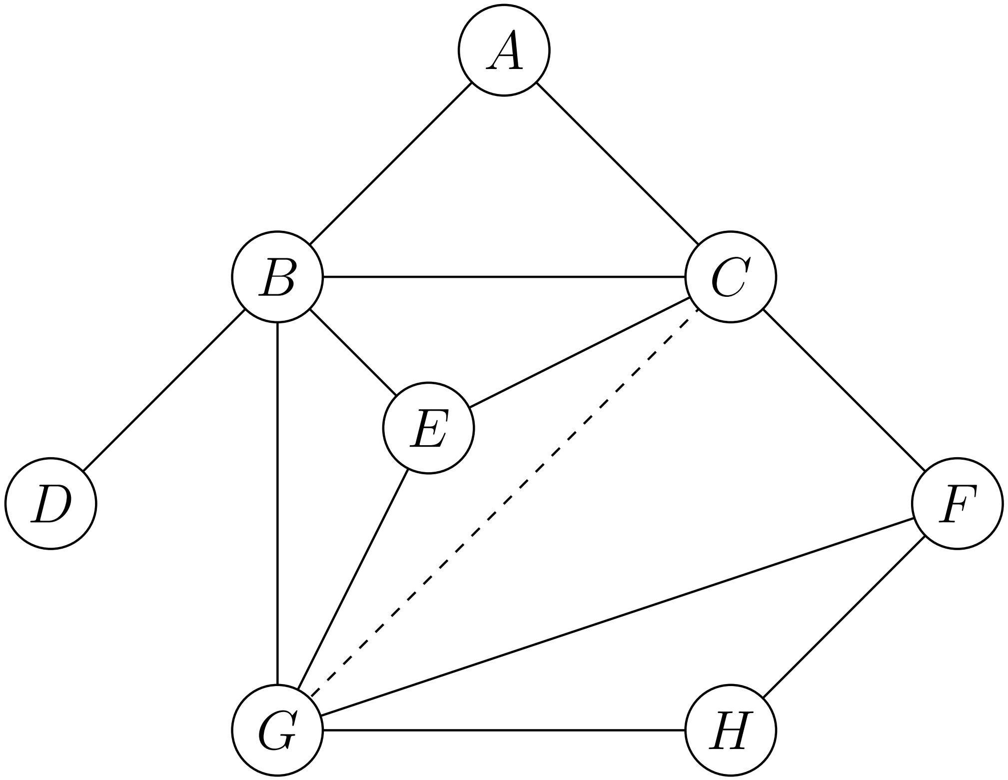
1. Moralize the graph.

# Example: Step 1: Find a Join-Tree



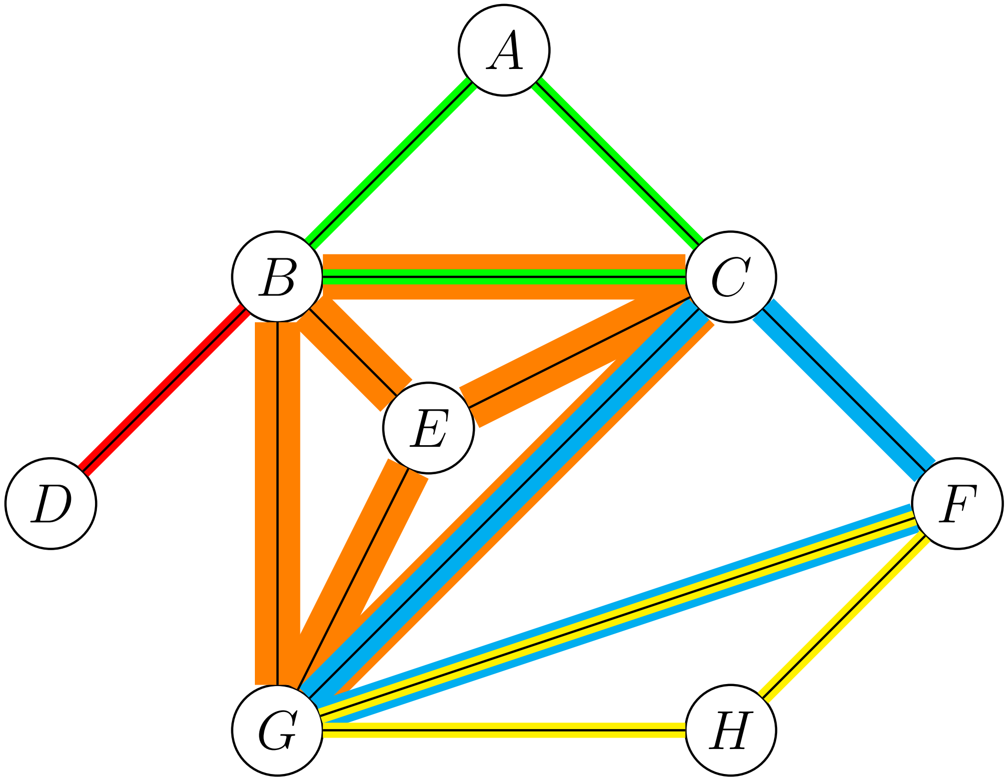

**Join-Tree creation:**

1. Moralize the graph.
2. Not yet triangulated.




**Join-Tree creation:**

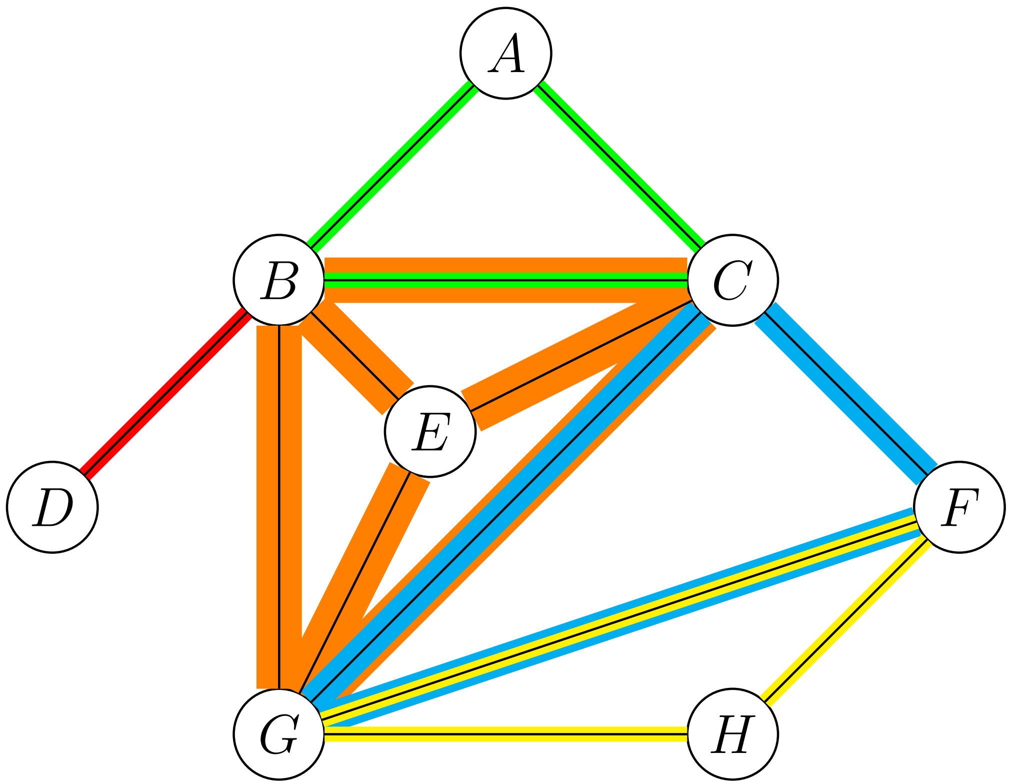
1. Moralize the graph.
2. Triangulate the graph.




**Join-Tree creation:**

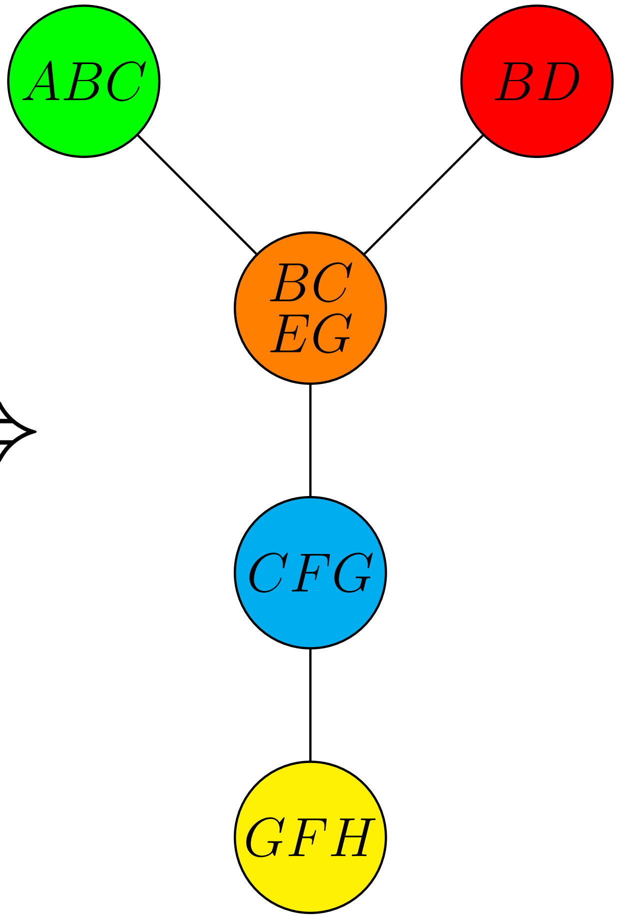
1. Moralize the graph.
2. Triangulate the graph.
3. Identify the maximal cliques.

# Example: Step 1: Find a Join-Tree



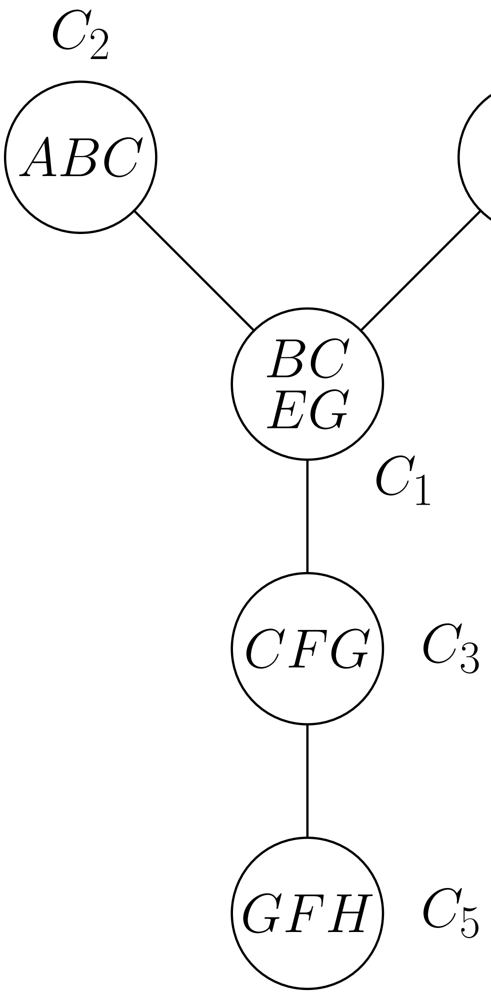

Example Bayesian network

$\Rightarrow$




One of the join trees

# Example: Step 2: Specify the Factor Potentials




Decomposition of $P(A, B, C, D, E, F, G, H)$:

$$\begin{aligned} P(a, b, c, d, e, f, g, h) &= \prod_{i=1}^{5} \Psi_i(c_i) \\ &= \Psi_1(b, c, e, g) \cdot \Psi_2(a, b, c) \\ &\quad \cdot \Psi_3(c, f, g) \cdot \Psi_4(b, d) \\ &\qquad \cdot \Psi_5(g, f, h) \end{aligned}$$

**Where to get the factor potentials from?**

# Example: Step 2: Specify the Factor Potentials

As long as the factor potentials multiply together as on the previous slide, we are free to choose them.

**Option 1:** A factor potential of clique $C_i$ is the product of all conditional probabilities of all node families properly contained in $C_i$:

$$\Psi_i(c_i) = 1 \cdot \prod_{\substack{\{X_i\} \cup Y_i \subseteq C_i \wedge \\ \text{parents}(X_i) = Y_i}} P(x_i \mid y_i)$$

The 1 stresses that if no node family satisfies the product condition, we assign a constant 1 to the potential.

**Option 2:** Choose potentials from the decomposition formula:

$$P(\bigcup_{i=1}^{n} C_i) = \frac{\prod_{i=1}^{n} P(C_i)}{\prod_{j=1}^{m} P(S_j)}$$

**Option 1:** Factor potentials according to the conditional distributions of the node families of the underlying Bayesian network:

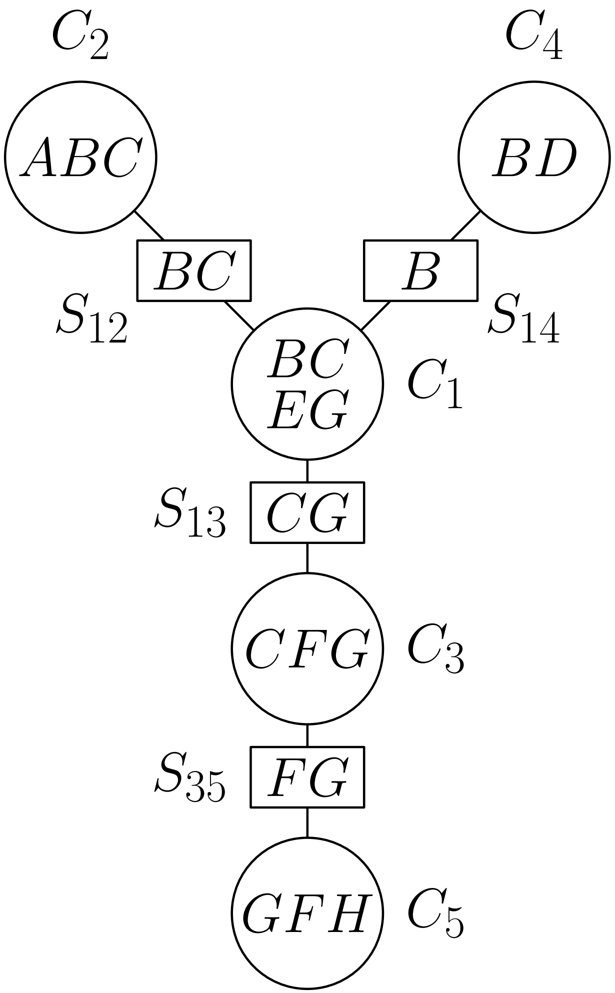
$$\begin{aligned}
\Psi_1(b,c,e,g) &= P(e \mid b,c) \cdot P(g \mid e,b) \\
\Psi_2(a,b,c) &= P(b \mid a) \cdot P(c \mid a) \cdot P(a) \\
\Psi_3(c,f,g) &= P(f \mid c) \\
\Psi_4(b,d) &= P(d \mid b) \\
\Psi_5(g,f,h) &= P(h \mid g,f)
\end{aligned}$$

(This assignment of factor potentials is used in this example.)

**Option 2:** Factor potentials chosen from the join-tree decomposition:

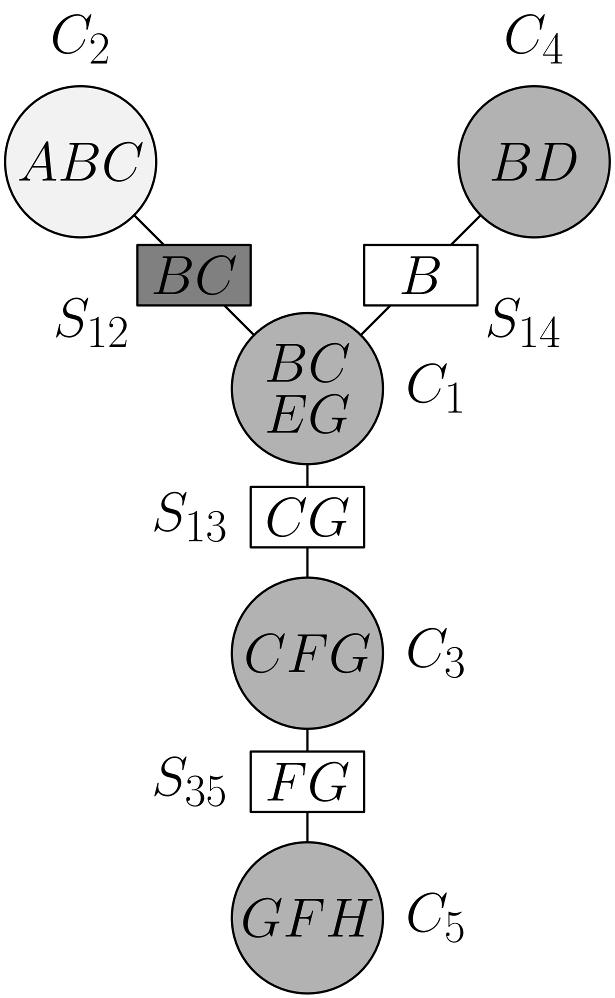
$$\begin{aligned}
\Psi_1(b,c,e,g) &= P(b,e \mid c,g) \\
\Psi_2(a,b,c) &= P(a \mid b,c) \\
\Psi_3(c,f,g) &= P(c \mid f,g) \\
\Psi_4(b,d) &= P(d \mid b) \\
\Psi_5(g,f,h) &= P(h,g,f)
\end{aligned}$$

# Example: Closer Look on Option 2: Separation in a Join-Tree




Encoded independence statements:

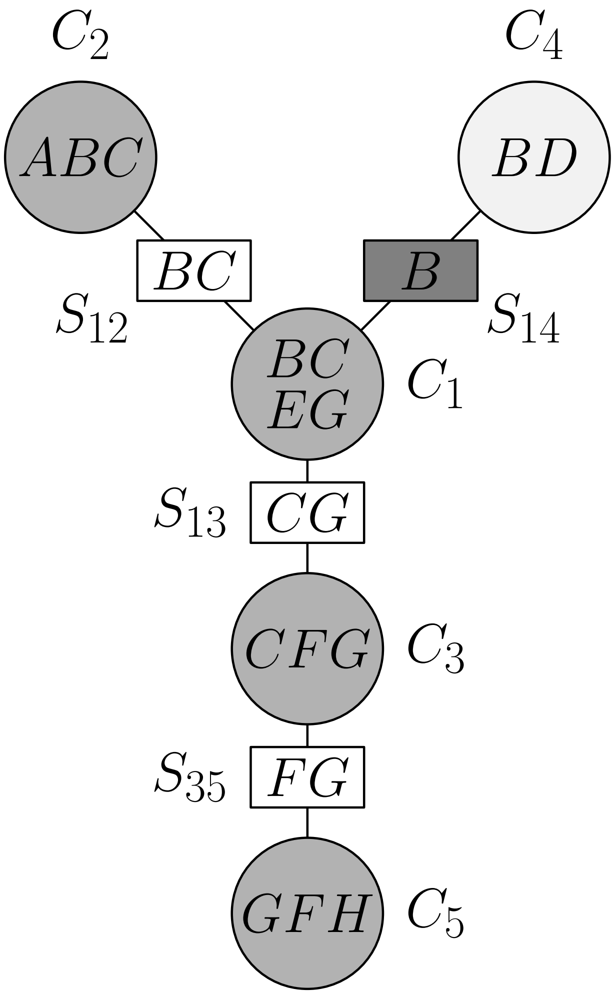
Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.




Encoded independence statements:

Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.

$$A \perp\!\!\!\perp D, E, F, G, H \mid B, C$$

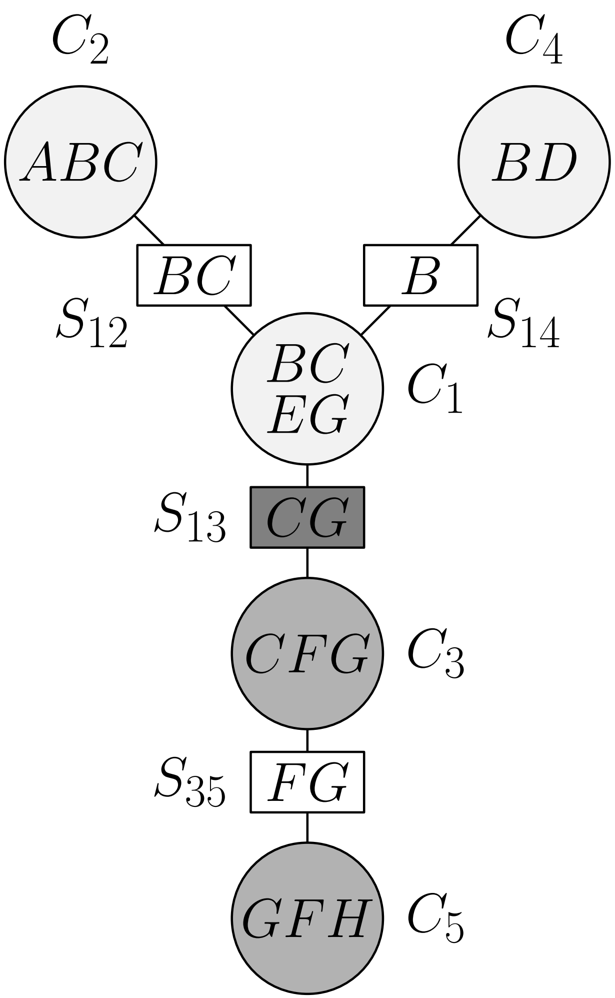



Encoded independence statements:

Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.

$$A \perp\!\!\!\perp D, E, F, G, H \mid B, C$$
$$D \perp\!\!\!\perp A, C, E, F, G, H \mid B$$

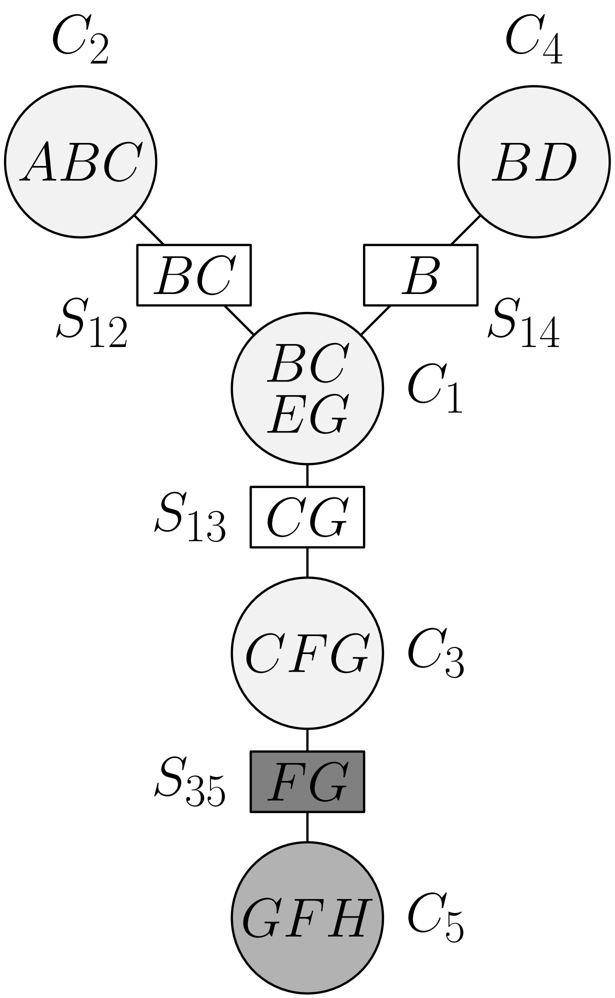
# Example: Closer Look on Option 2: Separation in a Join-Tree




Encoded independence statements:

Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.

$$A \perp\!\!\!\perp D, E, F, G, H \mid B, C$$
$$D \perp\!\!\!\perp A, C, E, F, G, H \mid B$$
$$A, B, E, D \perp\!\!\!\perp F, H \mid G, C$$




Encoded independence statements:

Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.

$$A \perp\!\!\!\perp D, E, F, G, H \mid B, C$$
$$D \perp\!\!\!\perp A, C, E, F, G, H \mid B$$
$$A, B, E, D \perp\!\!\!\perp F, H \mid G, C$$
$$H \perp\!\!\!\perp A, B, C, D, E \mid F, G$$

# Example: Closer Look on Option 2: Decomposition

The four separation statements translate into the following independence statements:

$$\begin{aligned}
A \perp\!\!\!\perp D, E, F, G, H \mid B, C &\Leftrightarrow P(A \mid B, C, D, E, F, G, H) &= P(A \mid B, C) \\
D \perp\!\!\!\perp A, C, E, F, G, H \mid B &\Rightarrow P(D \mid B, C, E, F, G, H) &= P(D \mid B) \\
A, B, E, D \perp\!\!\!\perp F, H \mid G, C &\Rightarrow P(B, E \mid G, C, F, H) &= P(B, E \mid G, C) \\
H \perp\!\!\!\perp A, B, C, D, E \mid F, G &\Rightarrow P(C \mid F, G, H) &= P(C \mid F, G)
\end{aligned}$$

According to the chain rule we always have the following relation:

$$\begin{aligned}
P(A, B, C, D, E, F, G, H) = \; & P(A \mid B, C, D, E, F, G, H) \cdot \\
& P(D \mid B, C, E, F, G, H) \cdot \\
& P(B, E \mid C, F, G, H) \cdot \\
& P(C \mid F, G, H) \cdot \\
& P(F, G, H)
\end{aligned}$$

## Example: Closer Look on Option 2: Decomposition

The four separation statements translate into the following independence statements:

$$
\begin{aligned}
A \perp\!\!\!\perp D, E, F, G, H \mid B, C \quad &\Leftrightarrow P(A \mid B, C, D, E, F, G, H) &= P(A \mid B, C) \\
D \perp\!\!\!\perp A, C, E, F, G, H \mid B \quad &\Rightarrow P(D \mid B, C, E, F, G, H) &= P(D \mid B) \\
A, B, E, D \perp\!\!\!\perp F, H \mid G, C \quad &\Rightarrow P(B, E \mid G, C, F, H) &= P(B, E \mid G, C) \\
H \perp\!\!\!\perp A, B, C, D, E \mid F, G \quad &\Rightarrow P(C \mid F, G, H) &= P(C \mid F, G)
\end{aligned}
$$

Exploiting the above independencies yields:

$$
\begin{aligned}
P(A, B, C, D, E, F, G, H) \; = \; & P(A \mid B, C) \cdot \\
& P(D \mid B) \cdot \\
& P(B, E \mid C, G) \cdot \\
& P(C \mid F, G) \cdot \\
& P(F, G, H)
\end{aligned}
$$

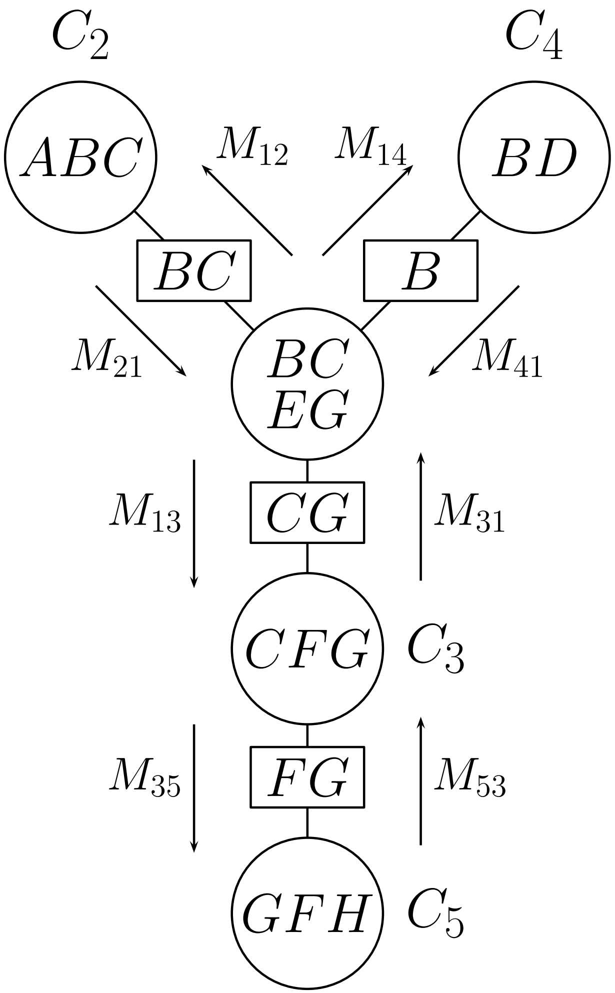
The four separation statements translate into the following independence statements:

$$\begin{aligned}
A \perp\!\!\!\perp D, E, F, G, H \mid B, C &\Leftrightarrow P(A \mid B, C, D, E, F, G, H) &= P(A \mid B, C) \\
D \perp\!\!\!\perp A, C, E, F, G, H \mid B &\Rightarrow P(D \mid B, C, E, F, G, H) &= P(D \mid B) \\
A, B, E, D \perp\!\!\!\perp F, H \mid G, C &\Rightarrow P(B, E \mid G, C, F, H) &= P(B, E \mid G, C) \\
H \perp\!\!\!\perp A, B, C, D, E \mid F, G &\Rightarrow P(C \mid F, G, H) &= P(C \mid F, G)
\end{aligned}$$

Getting rid of the conditions results in the final decomposition equation:

$$\begin{aligned}
P(A, B, C, D, E, F, G, H) &= P(A \mid B, C) P(D \mid B) P(B, E \mid C, G) P(C \mid F, G) P(F, G, H) \\
&= \frac{P(A, B, C) P(D, B) P(B, E, C, G) P(C, F, G) P(F, G, H)}{P(B, C) P(B) P(C, G) P(F, G)} \\
&= \frac{P(C_1) P(C_2) P(C_3) P(C_4) P(C_5)}{P(S_{12}) P(S_{14}) P(S_{13}) P(S_{35})}
\end{aligned}$$

# Example: Step 3: Messages to be sent for Propagation




According to the join-tree propagation algorithm, the probability distributions of all clique instantiations $c_i$ is calculated as follows:

$$P(c_i) \;\propto\; \Psi_i(c_i) \prod_{j=1}^{q} M_{ji}(s_{ij})$$

Spelt out for our example, we get:

$$\begin{aligned}
P(c_1) &= P(b,c,e,g) &&= \Psi_1(b,c,e,g) &&\cdot M_{21}(b,c) \cdot M_{31}(c,g) \cdot M_{41}(b)\\
P(c_2) &= P(a,b,c) &&\propto \Psi_2(a,b,c) &&\cdot M_{12}(b,c)\\
P(c_3) &= P(c,f,g) &&\propto \Psi_3(c,f,g) &&\cdot M_{13}(c,g) \cdot M_{53}(f,g)\\
P(c_4) &= P(b,d) &&\propto \Psi_4(b,d) &&\cdot M_{14}(b)\\
P(c_5) &= P(f,g,h) &&\propto \Psi_5(f,g,h) &&\cdot M_{35}(f,g)
\end{aligned}$$

The $\propto$-symbol indicates that the right-hand side may not add up to one. In that case we just normalize.

# Example: Step 3: Message Computation Order

The structure of the join-tree imposes a partial ordering according to which the messages need to be computed:

$$M_{41}(b) = \sum_{d} \Psi_4(b, d)$$

$$M_{53}(f, g) = \sum_{h} \Psi_5(f, g, h)$$

$$M_{21}(b, c) = \sum_{a} \Psi_2(a, b, c)$$

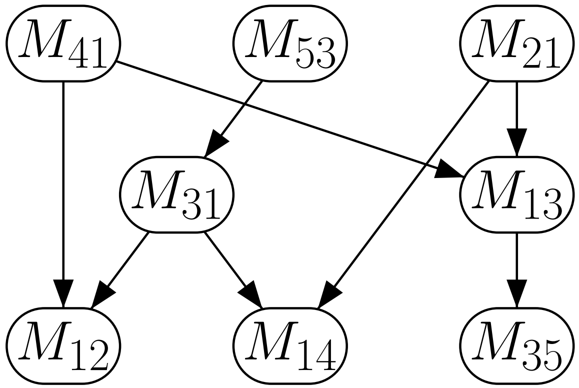
$$M_{31}(c, g) = \sum_{f} \Psi_3(c, f, g) M_{53}(f, g)$$

$$M_{13}(c, g) = \sum_{b,e} \Psi_1(b, c, e, g) M_{21}(b, c) M_{41}(b)$$

$$M_{12}(b, c) = \sum_{e,g} \Psi_2(b, c, e, g) M_{31}(c, g) M_{41}(b)$$

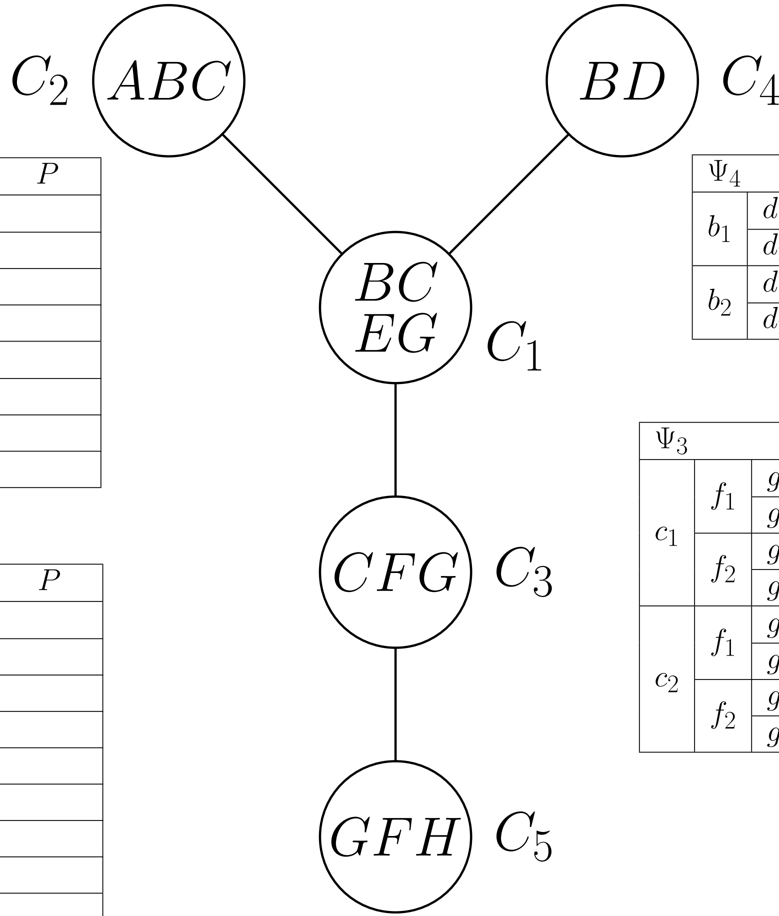
$$M_{14}(b) = \sum_{c,e,g} \Psi_1(b, c, e, g) M_{21}(b, c) M_{31}(c, g)$$

$$M_{35}(f, g) = \sum_{c} \Psi_3(c, f, g) M_{13}(c, g)$$




Arrows represent is-needed-for relations. Messages on the same level can be computed in any order. Messages are computed levelwise from top to bottom.

# Example: Step 3: Initialization (Potential Layouts)




| $\Psi_2$ | | | | $P$ |
|---|---|---|---|---|
| $a_1$ | $b_1$ | $c_1$ | | |
| | | $c_2$ | | |
| | $b_2$ | $c_1$ | | |
| | | $c_2$ | | |
| $a_2$ | $b_1$ | $c_1$ | | |
| | | $c_2$ | | |
| | $b_2$ | $c_1$ | | |
| | | $c_2$ | | |

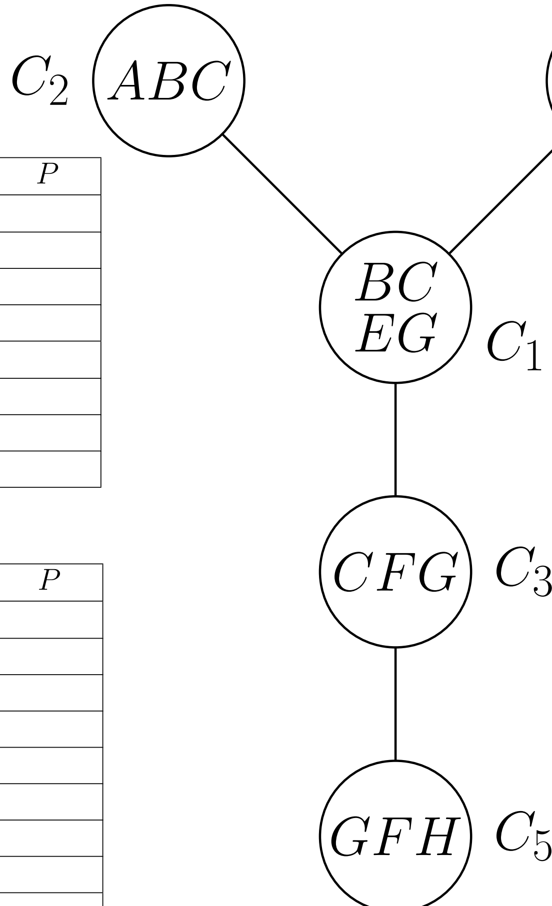
| $\Psi_4$ | | | $P$ |
|---|---|---|---|
| $b_1$ | $d_1$ | | |
| | $d_2$ | | |
| $b_2$ | $d_1$ | | |
| | $d_2$ | | |

| $\Psi_3$ | | | | $P$ |
|---|---|---|---|---|
| $c_1$ | $f_1$ | $g_1$ | | |
| | | $g_2$ | | |
| | $f_2$ | $g_1$ | | |
| | | $g_2$ | | |
| $c_2$ | $f_1$ | $g_1$ | | |
| | | $g_2$ | | |
| | $f_2$ | $g_1$ | | |
| | | $g_2$ | | |

| $\Psi_1$ | | | | | $P$ |
|---|---|---|---|---|---|
| $b_1$ | $c_1$ | $e_1$ | $g_1$ | | |
| | | | $g_2$ | | |
| | | $e_2$ | $g_1$ | | |
| | | | $g_2$ | | |
| | $c_2$ | $e_1$ | $g_1$ | | |
| | | | $g_2$ | | |
| | | $e_2$ | $g_1$ | | |
| | | | $g_2$ | | |
| $b_2$ | $c_1$ | $e_1$ | $g_1$ | | |
| | | | $g_2$ | | |
| | | $e_2$ | $g_1$ | | |
| | | | $g_2$ | | |
| | $c_2$ | $e_1$ | $g_1$ | | |
| | | | $g_2$ | | |
| | | $e_2$ | $g_1$ | | |
| | | | $g_2$ | | |

| $\Psi_5$ | | | | $P$ |
|---|---|---|---|---|
| $f_1$ | $g_1$ | $h_1$ | | |
| | | $h_2$ | | |
| | $g_2$ | $h_1$ | | |
| | | $h_2$ | | |
| $f_2$ | $g_1$ | $h_1$ | | |
| | | $h_2$ | | |
| | $g_2$ | $h_1$ | | |
| | | $h_2$ | | |

# Example: Step 3: Initialization (Potential Values)



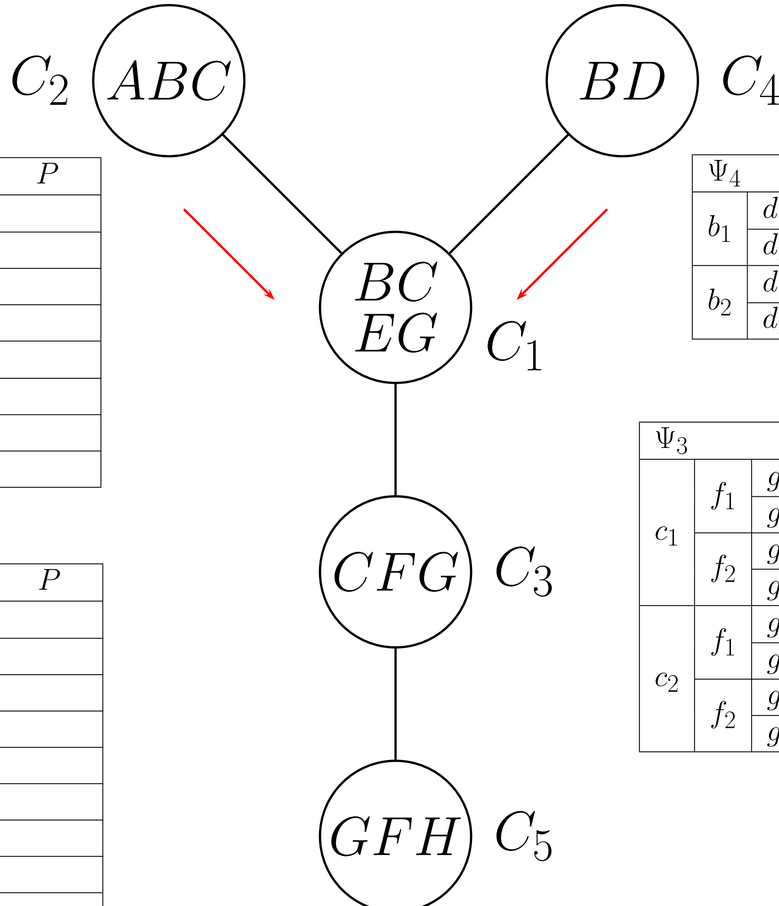

| $\Psi_2$ | | | | $P$ |
|---|---|---|---|---|
| $a_1$ | $b_1$ | $c_1$ | 0.036 | |
| | | $c_2$ | 0.084 | |
| | $b_2$ | $c_1$ | 0.144 | |
| | | $c_2$ | 0.336 | |
| $a_2$ | $b_1$ | $c_1$ | 0.028 | |
| | | $c_2$ | 0.012 | |
| | $b_2$ | $c_1$ | 0.252 | |
| | | $c_2$ | 0.108 | |

| $\Psi_4$ | | | $P$ |
|---|---|---|---|
| $b_1$ | $d_1$ | 0.4 | |
| | $d_2$ | 0.6 | |
| $b_2$ | $d_1$ | 0.7 | |
| | $d_2$ | 0.3 | |

| $\Psi_1$ | | | | | $P$ |
|---|---|---|---|---|---|
| $b_1$ | $c_1$ | $e_1$ | $g_1$ | 0.190 | |
| | | | $g_2$ | 0.010 | |
| | | $e_2$ | $g_1$ | 0.320 | |
| | | | $g_2$ | 0.480 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.380 | |
| | | | $g_2$ | 0.020 | |
| | | $e_2$ | $g_1$ | 0.240 | |
| | | | $g_2$ | 0.360 | |
| $b_2$ | $c_1$ | $e_1$ | $g_1$ | 0.210 | |
| | | | $g_2$ | 0.090 | |
| | | $e_2$ | $g_1$ | 0.350 | |
| | | | $g_2$ | 0.350 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.070 | |
| | | | $g_2$ | 0.030 | |
| | | $e_2$ | $g_1$ | 0.450 | |
| | | | $g_2$ | 0.450 | |

| $\Psi_3$ | | | | $P$ |
|---|---|---|---|---|
| $c_1$ | $f_1$ | $g_1$ | 0.1 | |
| | | $g_2$ | 0.1 | |
| | $f_2$ | $g_1$ | 0.9 | |
| | | $g_2$ | 0.9 | |
| $c_2$ | $f_1$ | $g_1$ | 0.4 | |
| | | $g_2$ | 0.4 | |
| | $f_2$ | $g_1$ | 0.6 | |
| | | $g_2$ | 0.6 | |

| $\Psi_5$ | | | | $P$ |
|---|---|---|---|---|
| $f_1$ | $g_1$ | $h_1$ | 0.2 | |
| | | $h_2$ | 0.8 | |
| | $g_2$ | $h_1$ | 0.5 | |
| | | $h_2$ | 0.5 | |
| $f_2$ | $g_1$ | $h_1$ | 0.4 | |
| | | $h_2$ | 0.6 | |
| | $g_2$ | $h_1$ | 0.7 | |
| | | $h_2$ | 0.3 | |




$$M_{21} = \big(\overset{b_1,c_1}{0.06}, \overset{b_1,c_2}{0.10}, \overset{b_2,c_1}{0.40}, \overset{b_2,c_2}{0.44}\big)$$

$$M_{41} = \big(\overset{b_1}{1}, \overset{b_2}{1}\big)$$

| $\Psi_2$ | | | | $P$ |
|---|---|---|---|---|
| $a_1$ | $b_1$ | $c_1$ | 0.036 | |
| | | $c_2$ | 0.084 | |
| | $b_2$ | $c_1$ | 0.144 | |
| | | $c_2$ | 0.336 | |
| $a_2$ | $b_1$ | $c_1$ | 0.028 | |
| | | $c_2$ | 0.012 | |
| | $b_2$ | $c_1$ | 0.252 | |
| | | $c_2$ | 0.108 | |

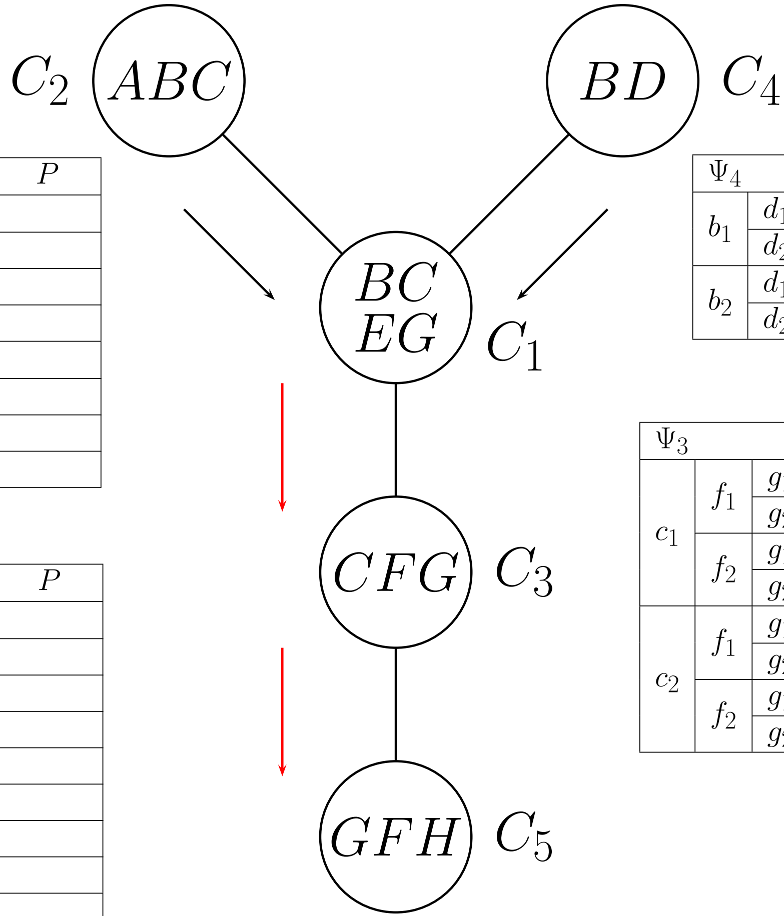
| $\Psi_4$ | | | $P$ |
|---|---|---|---|
| $b_1$ | $d_1$ | 0.4 | |
| | $d_2$ | 0.6 | |
| $b_2$ | $d_1$ | 0.7 | |
| | $d_2$ | 0.3 | |

| $\Psi_1$ | | | | | $P$ |
|---|---|---|---|---|---|
| $b_1$ | $c_1$ | $e_1$ | $g_1$ | 0.190 | |
| | | | $g_2$ | 0.010 | |
| | | $e_2$ | $g_1$ | 0.320 | |
| | | | $g_2$ | 0.480 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.380 | |
| | | | $g_2$ | 0.020 | |
| | | $e_2$ | $g_1$ | 0.240 | |
| | | | $g_2$ | 0.360 | |
| $b_2$ | $c_1$ | $e_1$ | $g_1$ | 0.210 | |
| | | | $g_2$ | 0.090 | |
| | | $e_2$ | $g_1$ | 0.350 | |
| | | | $g_2$ | 0.350 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.070 | |
| | | | $g_2$ | 0.030 | |
| | | $e_2$ | $g_1$ | 0.450 | |
| | | | $g_2$ | 0.450 | |

| $\Psi_3$ | | | | $P$ |
|---|---|---|---|---|
| $c_1$ | $f_1$ | $g_1$ | 0.1 | |
| | | $g_2$ | 0.1 | |
| | $f_2$ | $g_1$ | 0.9 | |
| | | $g_2$ | 0.9 | |
| $c_2$ | $f_1$ | $g_1$ | 0.4 | |
| | | $g_2$ | 0.4 | |
| | $f_2$ | $g_1$ | 0.6 | |
| | | $g_2$ | 0.6 | |

| $\Psi_5$ | | | | $P$ |
|---|---|---|---|---|
| $f_1$ | $g_1$ | $h_1$ | 0.2 | |
| | | $h_2$ | 0.8 | |
| | $g_2$ | $h_1$ | 0.5 | |
| | | $h_2$ | 0.5 | |
| $f_2$ | $g_1$ | $h_1$ | 0.4 | |
| | | $h_2$ | 0.6 | |
| | $g_2$ | $h_1$ | 0.7 | |
| | | $h_2$ | 0.3 | |

# Example: Step 3: Initialization (Sending Messages)




$$M_{21} = \left(\overset{b_1,c_1}{0.06}, \overset{b_1,c_2}{0.10}, \overset{b_2,c_1}{0.40}, \overset{b_2,c_2}{0.44}\right)$$

$$M_{41} = \left(\overset{b_1}{1}, \overset{b_2}{1}\right)$$

$$M_{13} = \left(\overset{c_1,g_1}{0.254}, \overset{c_1,g_2}{0.206}, \overset{c_2,g_1}{0.290}, \overset{c_2,g_2}{0.250}\right)$$

$$M_{35} = \left(\overset{f_1,g_1}{0.14}, \overset{f_1,g_2}{0.12}, \overset{f_2,g_1}{0.40}, \overset{f_2,g_2}{0.33}\right)$$

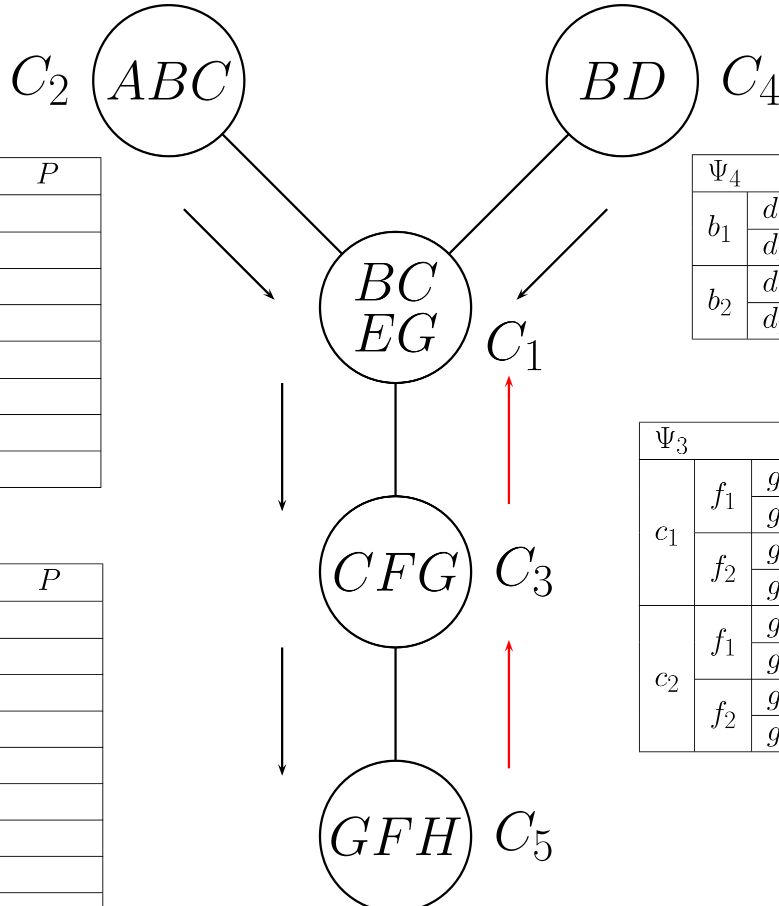
| $\Psi_2$ | | | | $P$ |
|---|---|---|---|---|
| $a_1$ | $b_1$ | $c_1$ | 0.036 | |
| | | $c_2$ | 0.084 | |
| | $b_2$ | $c_1$ | 0.144 | |
| | | $c_2$ | 0.336 | |
| $a_2$ | $b_1$ | $c_1$ | 0.028 | |
| | | $c_2$ | 0.012 | |
| | $b_2$ | $c_1$ | 0.252 | |
| | | $c_2$ | 0.108 | |

| $\Psi_4$ | | | $P$ |
|---|---|---|---|
| $b_1$ | $d_1$ | 0.4 | |
| | $d_2$ | 0.6 | |
| $b_2$ | $d_1$ | 0.7 | |
| | $d_2$ | 0.3 | |

| $\Psi_3$ | | | | $P$ |
|---|---|---|---|---|
| $c_1$ | $f_1$ | $g_1$ | 0.1 | |
| | | $g_2$ | 0.1 | |
| | $f_2$ | $g_1$ | 0.9 | |
| | | $g_2$ | 0.9 | |
| $c_2$ | $f_1$ | $g_1$ | 0.4 | |
| | | $g_2$ | 0.4 | |
| | $f_2$ | $g_1$ | 0.6 | |
| | | $g_2$ | 0.6 | |

| $\Psi_1$ | | | | | $P$ |
|---|---|---|---|---|---|
| $b_1$ | $c_1$ | $e_1$ | $g_1$ | 0.190 | |
| | | | $g_2$ | 0.010 | |
| | | $e_2$ | $g_1$ | 0.320 | |
| | | | $g_2$ | 0.480 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.380 | |
| | | | $g_2$ | 0.020 | |
| | | $e_2$ | $g_1$ | 0.240 | |
| | | | $g_2$ | 0.360 | |
| $b_2$ | $c_1$ | $e_1$ | $g_1$ | 0.210 | |
| | | | $g_2$ | 0.090 | |
| | | $e_2$ | $g_1$ | 0.350 | |
| | | | $g_2$ | 0.350 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.070 | |
| | | | $g_2$ | 0.030 | |
| | | $e_2$ | $g_1$ | 0.450 | |
| | | | $g_2$ | 0.450 | |

| $\Psi_5$ | | | | $P$ |
|---|---|---|---|---|
| $f_1$ | $g_1$ | $h_1$ | 0.2 | |
| | | $h_2$ | 0.8 | |
| | $g_2$ | $h_1$ | 0.5 | |
| | | $h_2$ | 0.5 | |
| $f_2$ | $g_1$ | $h_1$ | 0.4 | |
| | | $h_2$ | 0.6 | |
| | $g_2$ | $h_1$ | 0.7 | |
| | | $h_2$ | 0.3 | |




$$M_{21} = \left(\overset{b_1,c_1}{0.06}, \overset{b_1,c_2}{0.10}, \overset{b_2,c_1}{0.40}, \overset{b_2,c_2}{0.44}\right)$$

$$M_{41} = \left(\overset{b_1}{1}, \overset{b_2}{1}\right)$$

$$M_{13} = \left(\overset{c_1,g_1}{0.254}, \overset{c_1,g_2}{0.206}, \overset{c_2,g_1}{0.290}, \overset{c_2,g_2}{0.250}\right)$$

$$M_{35} = \left(\overset{f_1,g_1}{0.14}, \overset{f_1,g_2}{0.12}, \overset{f_2,g_1}{0.40}, \overset{f_2,g_2}{0.33}\right)$$

$$M_{53} = \left(\overset{f_1,g_1}{1}, \overset{f_1,g_2}{1}, \overset{f_2,g_1}{1}, \overset{f_2,g_2}{1}\right)$$

$$M_{31} = \left(\overset{c_1,g_1}{1}, \overset{c_1,g_2}{1}, \overset{c_2,g_1}{1}, \overset{c_2,g_2}{1}\right)$$

| $\Psi_2$ | | | | $P$ |
|---|---|---|---|---|
| $a_1$ | $b_1$ | $c_1$ | 0.036 | |
| | | $c_2$ | 0.084 | |
| | $b_2$ | $c_1$ | 0.144 | |
| | | $c_2$ | 0.336 | |
| $a_2$ | $b_1$ | $c_1$ | 0.028 | |
| | | $c_2$ | 0.012 | |
| | $b_2$ | $c_1$ | 0.252 | |
| | | $c_2$ | 0.108 | |

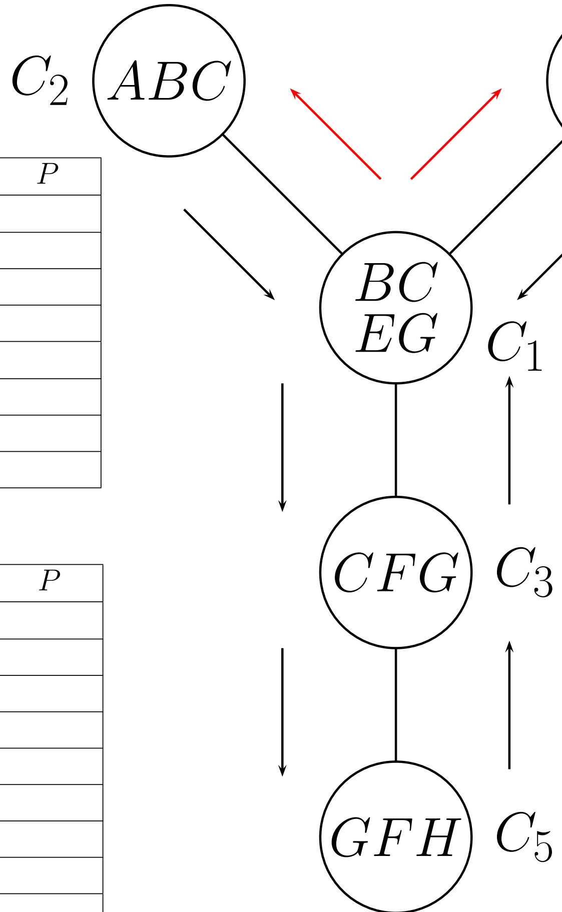
| $\Psi_4$ | | | $P$ |
|---|---|---|---|
| $b_1$ | $d_1$ | 0.4 | |
| | $d_2$ | 0.6 | |
| $b_2$ | $d_1$ | 0.7 | |
| | $d_2$ | 0.3 | |

| $\Psi_3$ | | | | $P$ |
|---|---|---|---|---|
| $c_1$ | $f_1$ | $g_1$ | 0.1 | |
| | | $g_2$ | 0.1 | |
| | $f_2$ | $g_1$ | 0.9 | |
| | | $g_2$ | 0.9 | |
| $c_2$ | $f_1$ | $g_1$ | 0.4 | |
| | | $g_2$ | 0.4 | |
| | $f_2$ | $g_1$ | 0.6 | |
| | | $g_2$ | 0.6 | |

| $\Psi_1$ | | | | | $P$ |
|---|---|---|---|---|---|
| $b_1$ | $c_1$ | $e_1$ | $g_1$ | 0.190 | |
| | | | $g_2$ | 0.010 | |
| | | $e_2$ | $g_1$ | 0.320 | |
| | | | $g_2$ | 0.480 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.380 | |
| | | | $g_2$ | 0.020 | |
| | | $e_2$ | $g_1$ | 0.240 | |
| | | | $g_2$ | 0.360 | |
| $b_2$ | $c_1$ | $e_1$ | $g_1$ | 0.210 | |
| | | | $g_2$ | 0.090 | |
| | | $e_2$ | $g_1$ | 0.350 | |
| | | | $g_2$ | 0.350 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.070 | |
| | | | $g_2$ | 0.030 | |
| | | $e_2$ | $g_1$ | 0.450 | |
| | | | $g_2$ | 0.450 | |

| $\Psi_5$ | | | | $P$ |
|---|---|---|---|---|
| $f_1$ | $g_1$ | $h_1$ | 0.2 | |
| | | $h_2$ | 0.8 | |
| | $g_2$ | $h_1$ | 0.5 | |
| | | $h_2$ | 0.5 | |
| $f_2$ | $g_1$ | $h_1$ | 0.4 | |
| | | $h_2$ | 0.6 | |
| | $g_2$ | $h_1$ | 0.7 | |
| | | $h_2$ | 0.3 | |

# Example: Step 3: Initialization (Sending Messages)




| $\Psi_2$ | | | | $P$ |
|---|---|---|---|---|
| $a_1$ | $b_1$ | $c_1$ | 0.036 | |
| | | $c_2$ | 0.084 | |
| | $b_2$ | $c_1$ | 0.144 | |
| | | $c_2$ | 0.336 | |
| $a_2$ | $b_1$ | $c_1$ | 0.028 | |
| | | $c_2$ | 0.012 | |
| | $b_2$ | $c_1$ | 0.252 | |
| | | $c_2$ | 0.108 | |

| $\Psi_4$ | | | $P$ |
|---|---|---|---|
| $b_1$ | $d_1$ | 0.4 | |
| | $d_2$ | 0.6 | |
| $b_2$ | $d_1$ | 0.7 | |
| | $d_2$ | 0.3 | |

| $\Psi_1$ | | | | | $P$ |
|---|---|---|---|---|---|
| $b_1$ | $c_1$ | $e_1$ | $g_1$ | 0.190 | |
| | | | $g_2$ | 0.010 | |
| | | $e_2$ | $g_1$ | 0.320 | |
| | | | $g_2$ | 0.480 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.380 | |
| | | | $g_2$ | 0.020 | |
| | | $e_2$ | $g_1$ | 0.240 | |
| | | | $g_2$ | 0.360 | |
| $b_2$ | $c_1$ | $e_1$ | $g_1$ | 0.210 | |
| | | | $g_2$ | 0.090 | |
| | | $e_2$ | $g_1$ | 0.350 | |
| | | | $g_2$ | 0.350 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.070 | |
| | | | $g_2$ | 0.030 | |
| | | $e_2$ | $g_1$ | 0.450 | |
| | | | $g_2$ | 0.450 | |

| $\Psi_3$ | | | | $P$ |
|---|---|---|---|---|
| $c_1$ | $f_1$ | $g_1$ | 0.1 | |
| | | $g_2$ | 0.1 | |
| | $f_2$ | $g_1$ | 0.9 | |
| | | $g_2$ | 0.9 | |
| $c_2$ | $f_1$ | $g_1$ | 0.4 | |
| | | $g_2$ | 0.4 | |
| | $f_2$ | $g_1$ | 0.6 | |
| | | $g_2$ | 0.6 | |

| $\Psi_5$ | | | | $P$ |
|---|---|---|---|---|
| $f_1$ | $g_1$ | $h_1$ | 0.2 | |
| | | $h_2$ | 0.8 | |
| | $g_2$ | $h_1$ | 0.5 | |
| | | $h_2$ | 0.5 | |
| $f_2$ | $g_1$ | $h_1$ | 0.4 | |
| | | $h_2$ | 0.6 | |
| | $g_2$ | $h_1$ | 0.7 | |
| | | $h_2$ | 0.3 | |

$$M_{21} = \left(\overset{b_1,c_1}{0.06}, \overset{b_1,c_2}{0.10}, \overset{b_2,c_1}{0.40}, \overset{b_2,c_2}{0.44}\right)$$

$$M_{41} = \left(\overset{b_1}{1}, \overset{b_2}{1}\right)$$

$$M_{13} = \left(\overset{c_1,g_1}{0.254}, \overset{c_1,g_2}{0.206}, \overset{c_2,g_1}{0.290}, \overset{c_2,g_2}{0.250}\right)$$

$$M_{35} = \left(\overset{f_1,g_1}{0.14}, \overset{f_1,g_2}{0.12}, \overset{f_2,g_1}{0.40}, \overset{f_2,g_2}{0.33}\right)$$

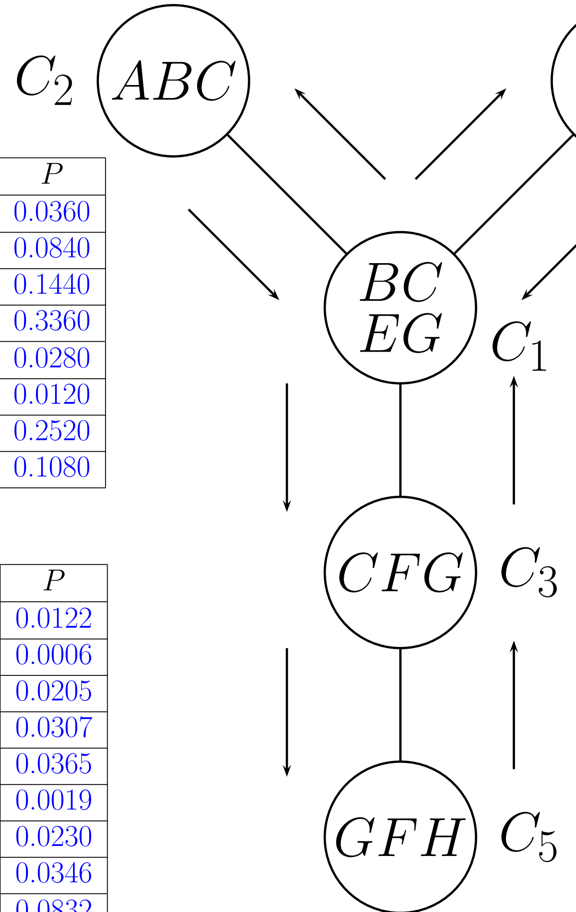
$$M_{53} = \left(\overset{f_1,g_1}{1}, \overset{f_1,g_2}{1}, \overset{f_2,g_1}{1}, \overset{f_2,g_2}{1}\right)$$

$$M_{31} = \left(\overset{c_1,g_1}{1}, \overset{c_1,g_2}{1}, \overset{c_2,g_1}{1}, \overset{c_2,g_2}{1}\right)$$

$$M_{12} = \left(\overset{b_1,c_1}{1}, \overset{b_1,c_2}{1}, \overset{b_2,c_1}{1}, \overset{b_2,c_2}{1}\right)$$

$$M_{14} = \left(\overset{b_1}{0.16}, \overset{b_2}{0.84}\right)$$

# Example: Step 3: Initialization Complete




| $\Psi_2$ | | | | $P$ |
|---|---|---|---|---|
| $a_1$ | $b_1$ | $c_1$ | 0.036 | 0.0360 |
| | | $c_2$ | 0.084 | 0.0840 |
| | $b_2$ | $c_1$ | 0.144 | 0.1440 |
| | | $c_2$ | 0.336 | 0.3360 |
| $a_2$ | $b_1$ | $c_1$ | 0.028 | 0.0280 |
| | | $c_2$ | 0.012 | 0.0120 |
| | $b_2$ | $c_1$ | 0.252 | 0.2520 |
| | | $c_2$ | 0.108 | 0.1080 |

| $\Psi_4$ | | | $P$ |
|---|---|---|---|
| $b_1$ | $d_1$ | 0.4 | 0.0640 |
| | $d_2$ | 0.6 | 0.0960 |
| $b_2$ | $d_1$ | 0.7 | 0.5880 |
| | $d_2$ | 0.3 | 0.2520 |

| $\Psi_1$ | | | | | $P$ |
|---|---|---|---|---|---|
| $b_1$ | $c_1$ | $e_1$ | $g_1$ | 0.190 | 0.0122 |
| | | | $g_2$ | 0.010 | 0.0006 |
| | | $e_2$ | $g_1$ | 0.320 | 0.0205 |
| | | | $g_2$ | 0.480 | 0.0307 |
| | $c_2$ | $e_1$ | $g_1$ | 0.380 | 0.0365 |
| | | | $g_2$ | 0.020 | 0.0019 |
| | | $e_2$ | $g_1$ | 0.240 | 0.0230 |
| | | | $g_2$ | 0.360 | 0.0346 |
| $b_2$ | $c_1$ | $e_1$ | $g_1$ | 0.210 | 0.0832 |
| | | | $g_2$ | 0.090 | 0.0356 |
| | | $e_2$ | $g_1$ | 0.350 | 0.1386 |
| | | | $g_2$ | 0.350 | 0.1386 |
| | $c_2$ | $e_1$ | $g_1$ | 0.070 | 0.0311 |
| | | | $g_2$ | 0.030 | 0.0133 |
| | | $e_2$ | $g_1$ | 0.450 | 0.1998 |
| | | | $g_2$ | 0.450 | 0.1998 |

| $\Psi_3$ | | | | $P$ |
|---|---|---|---|---|
| $c_1$ | $f_1$ | $g_1$ | 0.1 | 0.0254 |
| | | $g_2$ | 0.1 | 0.0206 |
| | $f_2$ | $g_1$ | 0.9 | 0.2290 |
| | | $g_2$ | 0.9 | 0.1850 |
| $c_2$ | $f_1$ | $g_1$ | 0.4 | 0.1162 |
| | | $g_2$ | 0.4 | 0.0998 |
| | $f_2$ | $g_1$ | 0.6 | 0.1742 |
| | | $g_2$ | 0.6 | 0.1498 |

| $\Psi_5$ | | | | $P$ |
|---|---|---|---|---|
| $f_1$ | $g_1$ | $h_1$ | 0.2 | 0.0283 |
| | | $h_2$ | 0.8 | 0.1133 |
| | $g_2$ | $h_1$ | 0.5 | 0.0602 |
| | | $h_2$ | 0.5 | 0.0602 |
| $f_2$ | $g_1$ | $h_1$ | 0.4 | 0.1613 |
| | | $h_2$ | 0.6 | 0.2419 |
| | $g_2$ | $h_1$ | 0.7 | 0.2344 |
| | | $h_2$ | 0.3 | 0.1004 |

$$M_{21} = \left(\overset{b_1,c_1}{0.06}, \overset{b_1,c_2}{0.10}, \overset{b_2,c_1}{0.40}, \overset{b_2,c_2}{0.44}\right)$$

$$M_{41} = \left(\overset{b_1}{1}, \overset{b_2}{1}\right)$$

$$M_{13} = \left(\overset{c_1,g_1}{0.254}, \overset{c_1,g_2}{0.206}, \overset{c_2,g_1}{0.290}, \overset{c_2,g_2}{0.250}\right)$$

$$M_{35} = \left(\overset{f_1,g_1}{0.14}, \overset{f_1,g_2}{0.12}, \overset{f_2,g_1}{0.40}, \overset{f_2,g_2}{0.33}\right)$$

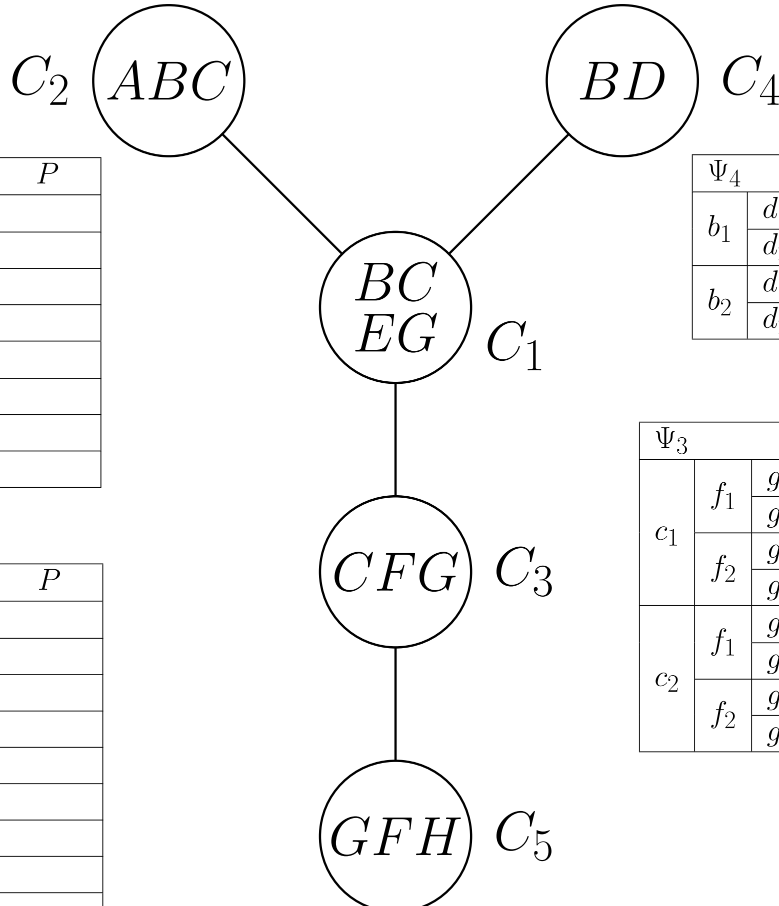
$$M_{53} = \left(\overset{f_1,g_1}{1}, \overset{f_1,g_2}{1}, \overset{f_2,g_1}{1}, \overset{f_2,g_2}{1}\right)$$

$$M_{31} = \left(\overset{c_1,g_1}{1}, \overset{c_1,g_2}{1}, \overset{c_2,g_1}{1}, \overset{c_2,g_2}{1}\right)$$

$$M_{12} = \left(\overset{b_1,c_1}{1}, \overset{b_1,c_2}{1}, \overset{b_2,c_1}{1}, \overset{b_2,c_2}{1}\right)$$

$$M_{14} = \left(\overset{b_1}{0.16}, \overset{b_2}{0.84}\right)$$

| $P$ | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
|---|---|---|---|---|---|---|---|---|
| $\cdot_1$ | **0.6000** | **0.1600** | **0.4600** | **0.6520** | **0.2144** | **0.2620** | **0.5448** | **0.4842** |
| $\cdot_2$ | **0.4000** | **0.8400** | **0.4500** | **0.3480** | **0.7856** | **0.7380** | **0.4552** | **0.5158** |



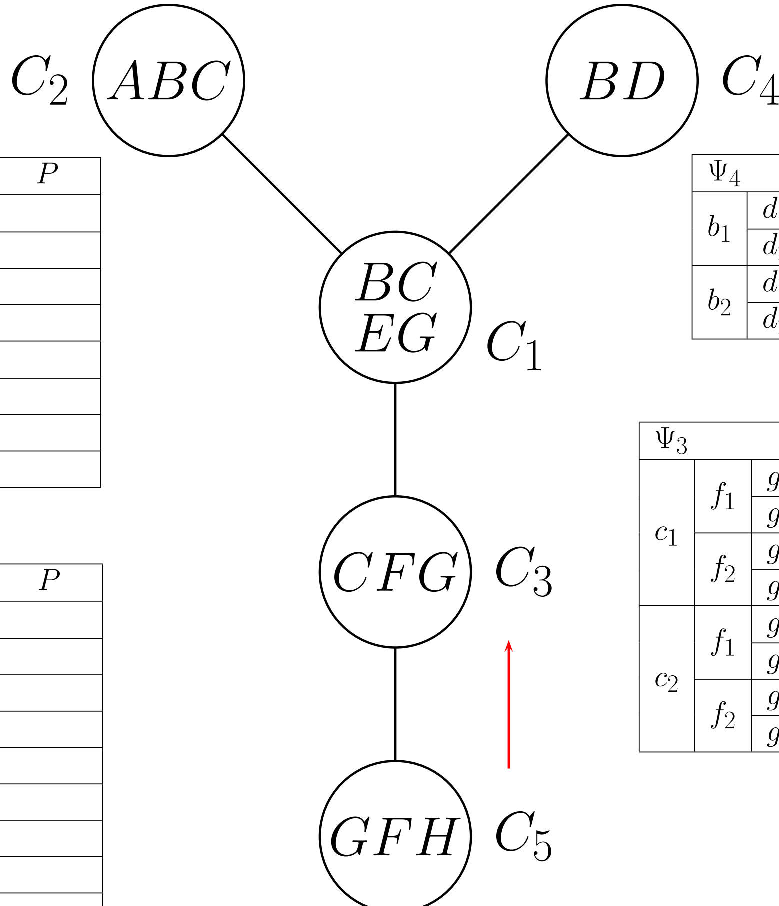

| $\Psi_2$ | | | | $P$ |
|---|---|---|---|---|
| $a_1$ | $b_1$ | $c_1$ | 0.036 | |
| | | $c_2$ | 0.084 | |
| | $b_2$ | $c_1$ | 0.144 | |
| | | $c_2$ | 0.336 | |
| $a_2$ | $b_1$ | $c_1$ | 0.028 | |
| | | $c_2$ | 0.012 | |
| | $b_2$ | $c_1$ | 0.252 | |
| | | $c_2$ | 0.108 | |

| $\Psi_4$ | | | $P$ |
|---|---|---|---|
| $b_1$ | $d_1$ | 0.4 | |
| | $d_2$ | 0.6 | |
| $b_2$ | $d_1$ | 0.7 | |
| | $d_2$ | 0.3 | |

| $\Psi_3$ | | | | $P$ |
|---|---|---|---|---|
| $c_1$ | $f_1$ | $g_1$ | 0.1 | |
| | | $g_2$ | 0.1 | |
| | $f_2$ | $g_1$ | 0.9 | |
| | | $g_2$ | 0.9 | |
| $c_2$ | $f_1$ | $g_1$ | 0.4 | |
| | | $g_2$ | 0.4 | |
| | $f_2$ | $g_1$ | 0.6 | |
| | | $g_2$ | 0.6 | |

| $\Psi_1$ | | | | | $P$ |
|---|---|---|---|---|---|
| $b_1$ | $c_1$ | $e_1$ | $g_1$ | 0.190 | |
| | | | $g_2$ | 0.010 | |
| | | $e_2$ | $g_1$ | 0.320 | |
| | | | $g_2$ | 0.480 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.380 | |
| | | | $g_2$ | 0.020 | |
| | | $e_2$ | $g_1$ | 0.240 | |
| | | | $g_2$ | 0.360 | |
| $b_2$ | $c_1$ | $e_1$ | $g_1$ | 0.210 | |
| | | | $g_2$ | 0.090 | |
| | | $e_2$ | $g_1$ | 0.350 | |
| | | | $g_2$ | 0.350 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.070 | |
| | | | $g_2$ | 0.030 | |
| | | $e_2$ | $g_1$ | 0.450 | |
| | | | $g_2$ | 0.450 | |

| $\Psi_5$ | | | | $P$ |
|---|---|---|---|---|
| $f_1$ | $g_1$ | $h_1$ | 0.2 | |
| | | $h_2$ | **0** | |
| | $g_2$ | $h_1$ | 0.5 | |
| | | $h_2$ | **0** | |
| $f_2$ | $g_1$ | $h_1$ | 0.4 | |
| | | $h_2$ | **0** | |
| | $g_2$ | $h_1$ | 0.7 | |
| | | $h_2$ | **0** | |




$$M_{53} = \begin{pmatrix} \overset{f_1,g_1}{0.2}, & \overset{f_1,g_2}{0.5}, & \overset{f_2,g_1}{0.4}, & \overset{f_2,g_2}{0.7} \end{pmatrix}$$

| $\Psi_2$ | | | | $P$ |
|---|---|---|---|---|
| $a_1$ | $b_1$ | $c_1$ | 0.036 | |
| | | $c_2$ | 0.084 | |
| | $b_2$ | $c_1$ | 0.144 | |
| | | $c_2$ | 0.336 | |
| $a_2$ | $b_1$ | $c_1$ | 0.028 | |
| | | $c_2$ | 0.012 | |
| | $b_2$ | $c_1$ | 0.252 | |
| | | $c_2$ | 0.108 | |

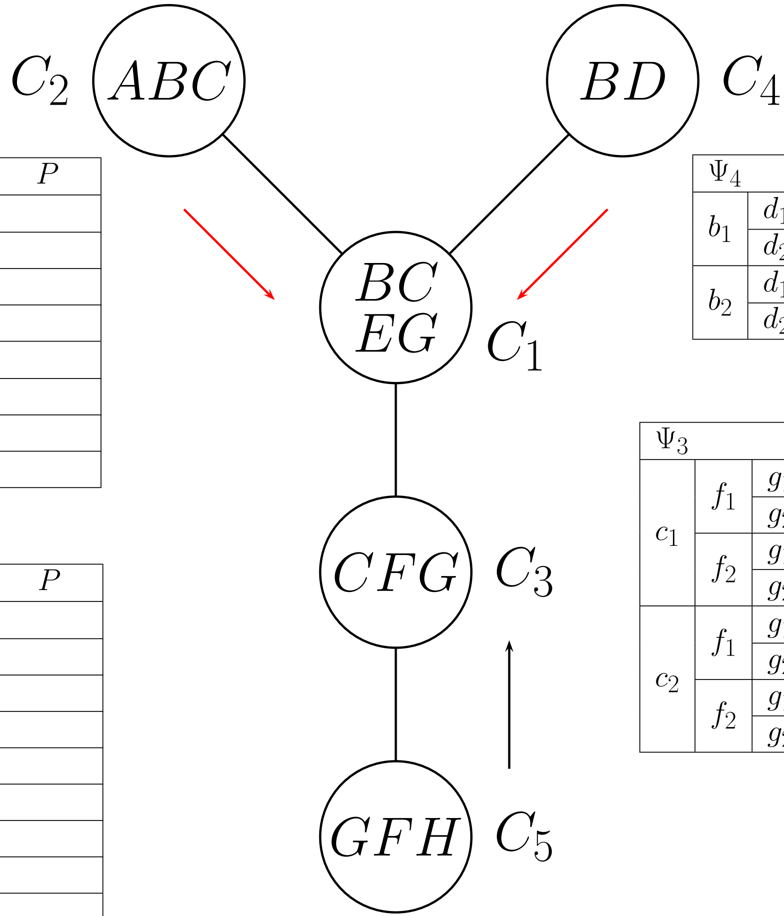
| $\Psi_4$ | | | $P$ |
|---|---|---|---|
| $b_1$ | $d_1$ | 0.4 | |
| | $d_2$ | 0.6 | |
| $b_2$ | $d_1$ | 0.7 | |
| | $d_2$ | 0.3 | |

| $\Psi_3$ | | | | $P$ |
|---|---|---|---|---|
| $c_1$ | $f_1$ | $g_1$ | 0.1 | |
| | | $g_2$ | 0.1 | |
| | $f_2$ | $g_1$ | 0.9 | |
| | | $g_2$ | 0.9 | |
| $c_2$ | $f_1$ | $g_1$ | 0.4 | |
| | | $g_2$ | 0.4 | |
| | $f_2$ | $g_1$ | 0.6 | |
| | | $g_2$ | 0.6 | |

| $\Psi_1$ | | | | | $P$ |
|---|---|---|---|---|---|
| $b_1$ | $c_1$ | $e_1$ | $g_1$ | 0.190 | |
| | | | $g_2$ | 0.010 | |
| | | $e_2$ | $g_1$ | 0.320 | |
| | | | $g_2$ | 0.480 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.380 | |
| | | | $g_2$ | 0.020 | |
| | | $e_2$ | $g_1$ | 0.240 | |
| | | | $g_2$ | 0.360 | |
| $b_2$ | $c_1$ | $e_1$ | $g_1$ | 0.210 | |
| | | | $g_2$ | 0.090 | |
| | | $e_2$ | $g_1$ | 0.350 | |
| | | | $g_2$ | 0.350 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.070 | |
| | | | $g_2$ | 0.030 | |
| | | $e_2$ | $g_1$ | 0.450 | |
| | | | $g_2$ | 0.450 | |

| $\Psi_5$ | | | | $P$ |
|---|---|---|---|---|
| $f_1$ | $g_1$ | $h_1$ | 0.2 | |
| | | $h_2$ | 0 | |
| | $g_2$ | $h_1$ | 0.5 | |
| | | $h_2$ | 0 | |
| $f_2$ | $g_1$ | $h_1$ | 0.4 | |
| | | $h_2$ | 0 | |
| | $g_2$ | $h_1$ | 0.7 | |
| | | $h_2$ | 0 | |

# Example: Step 4: Evidence $H = h_1$ (Sending Messages)




$$M_{53} = \left(\overset{f_1,g_1}{0.2}, \overset{f_1,g_2}{0.5}, \overset{f_2,g_1}{0.4}, \overset{f_2,g_2}{0.7}\right)$$

$$M_{21} = \left(\overset{b_1,c_1}{0.06}, \overset{b_1,c_2}{0.10}, \overset{b_2,c_1}{0.40}, \overset{b_2,c_2}{0.44}\right)$$

$$M_{41} = \left(\overset{b_1}{1}, \overset{b_2}{1}\right)$$

| $\Psi_2$ | | | | $P$ |
|---|---|---|---|---|
| $a_1$ | $b_1$ | $c_1$ | 0.036 | |
| | | $c_2$ | 0.084 | |
| | $b_2$ | $c_1$ | 0.144 | |
| | | $c_2$ | 0.336 | |
| $a_2$ | $b_1$ | $c_1$ | 0.028 | |
| | | $c_2$ | 0.012 | |
| | $b_2$ | $c_1$ | 0.252 | |
| | | $c_2$ | 0.108 | |

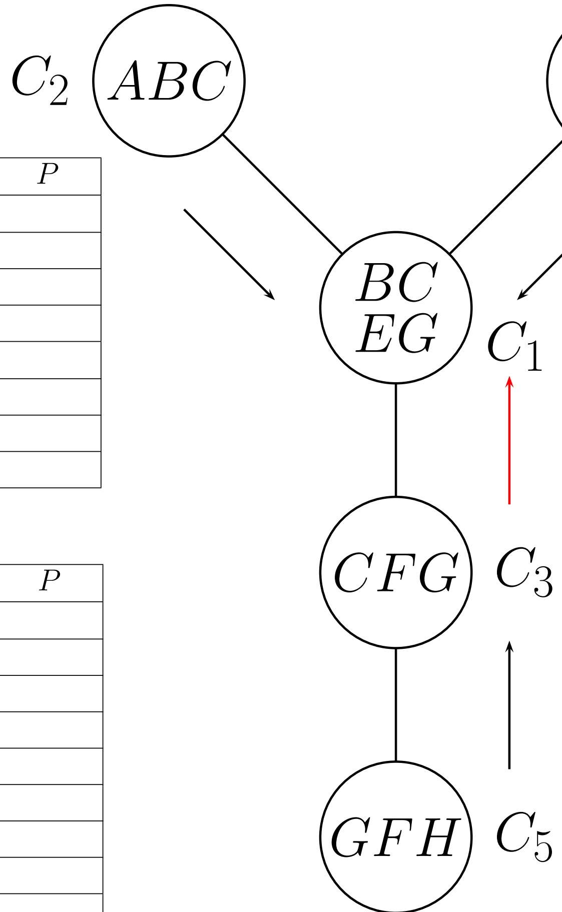
| $\Psi_4$ | | | $P$ |
|---|---|---|---|
| $b_1$ | $d_1$ | 0.4 | |
| | $d_2$ | 0.6 | |
| $b_2$ | $d_1$ | 0.7 | |
| | $d_2$ | 0.3 | |

| $\Psi_3$ | | | | $P$ |
|---|---|---|---|---|
| $c_1$ | $f_1$ | $g_1$ | 0.1 | |
| | | $g_2$ | 0.1 | |
| | $f_2$ | $g_1$ | 0.9 | |
| | | $g_2$ | 0.9 | |
| $c_2$ | $f_1$ | $g_1$ | 0.4 | |
| | | $g_2$ | 0.4 | |
| | $f_2$ | $g_1$ | 0.6 | |
| | | $g_2$ | 0.6 | |

| $\Psi_1$ | | | | | $P$ |
|---|---|---|---|---|---|
| $b_1$ | $c_1$ | $e_1$ | $g_1$ | 0.190 | |
| | | | $g_2$ | 0.010 | |
| | | $e_2$ | $g_1$ | 0.320 | |
| | | | $g_2$ | 0.480 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.380 | |
| | | | $g_2$ | 0.020 | |
| | | $e_2$ | $g_1$ | 0.240 | |
| | | | $g_2$ | 0.360 | |
| $b_2$ | $c_1$ | $e_1$ | $g_1$ | 0.210 | |
| | | | $g_2$ | 0.090 | |
| | | $e_2$ | $g_1$ | 0.350 | |
| | | | $g_2$ | 0.350 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.070 | |
| | | | $g_2$ | 0.030 | |
| | | $e_2$ | $g_1$ | 0.450 | |
| | | | $g_2$ | 0.450 | |

| $\Psi_5$ | | | | $P$ |
|---|---|---|---|---|
| $f_1$ | $g_1$ | $h_1$ | 0.2 | |
| | | $h_2$ | 0 | |
| | $g_2$ | $h_1$ | 0.5 | |
| | | $h_2$ | 0 | |
| $f_2$ | $g_1$ | $h_1$ | 0.4 | |
| | | $h_2$ | 0 | |
| | $g_2$ | $h_1$ | 0.7 | |
| | | $h_2$ | 0 | |

# Example: Step 4: Evidence $H = h_1$ (Sending Messages)




| $\Psi_2$ | | | | $P$ |
|---|---|---|---|---|
| $a_1$ | $b_1$ | $c_1$ | 0.036 | |
| | | $c_2$ | 0.084 | |
| | $b_2$ | $c_1$ | 0.144 | |
| | | $c_2$ | 0.336 | |
| $a_2$ | $b_1$ | $c_1$ | 0.028 | |
| | | $c_2$ | 0.012 | |
| | $b_2$ | $c_1$ | 0.252 | |
| | | $c_2$ | 0.108 | |

| $\Psi_4$ | | | $P$ |
|---|---|---|---|
| $b_1$ | $d_1$ | 0.4 | |
| | $d_2$ | 0.6 | |
| $b_2$ | $d_1$ | 0.7 | |
| | $d_2$ | 0.3 | |

$$M_{53} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 0.2, & 0.5, & 0.4, & 0.7 \end{pmatrix}$$

$$M_{21} = \begin{pmatrix} b_1,c_1 & b_1,c_2 & b_2,c_1 & b_2,c_2 \\ 0.06, & 0.10, & 0.40, & 0.44 \end{pmatrix}$$

$$M_{41} = \begin{pmatrix} b_1 & b_2 \\ 1, & 1 \end{pmatrix}$$

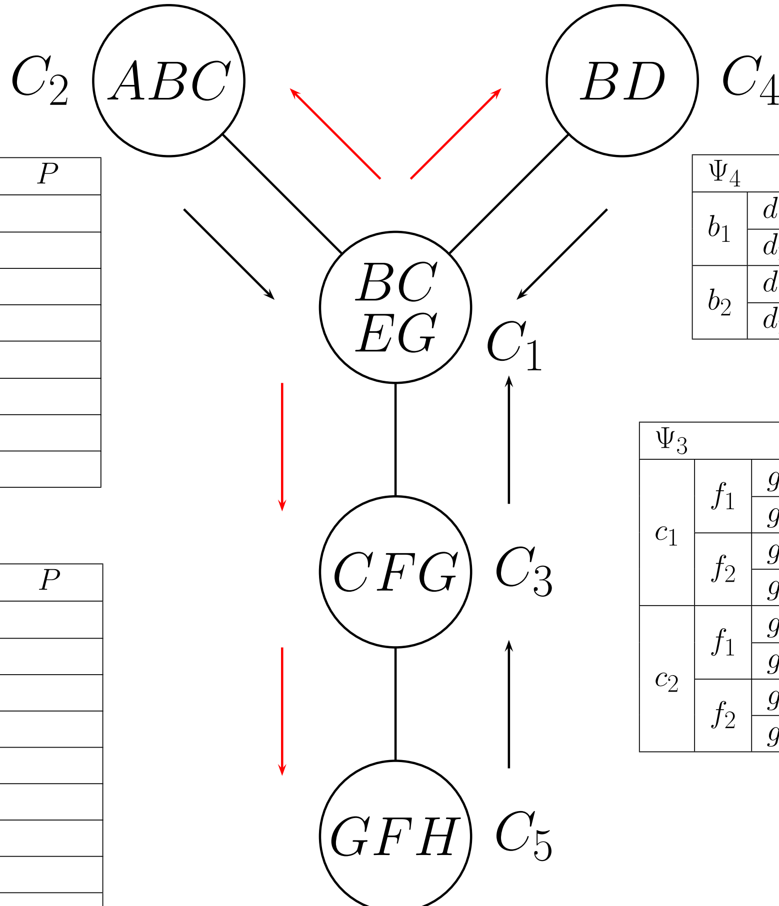
$$M_{31} = \begin{pmatrix} c_1,g_1 & c_1,g_2 & c_2,g_1 & c_2,g_2 \\ 0.38, & 0.68, & 0.32, & 0.62 \end{pmatrix}$$

| $\Psi_3$ | | | | $P$ |
|---|---|---|---|---|
| $c_1$ | $f_1$ | $g_1$ | 0.1 | |
| | | $g_2$ | 0.1 | |
| | $f_2$ | $g_1$ | 0.9 | |
| | | $g_2$ | 0.9 | |
| $c_2$ | $f_1$ | $g_1$ | 0.4 | |
| | | $g_2$ | 0.4 | |
| | $f_2$ | $g_1$ | 0.6 | |
| | | $g_2$ | 0.6 | |

| $\Psi_1$ | | | | | $P$ |
|---|---|---|---|---|---|
| $b_1$ | $c_1$ | $e_1$ | $g_1$ | 0.190 | |
| | | | $g_2$ | 0.010 | |
| | | $e_2$ | $g_1$ | 0.320 | |
| | | | $g_2$ | 0.480 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.380 | |
| | | | $g_2$ | 0.020 | |
| | | $e_2$ | $g_1$ | 0.240 | |
| | | | $g_2$ | 0.360 | |
| $b_2$ | $c_1$ | $e_1$ | $g_1$ | 0.210 | |
| | | | $g_2$ | 0.090 | |
| | | $e_2$ | $g_1$ | 0.350 | |
| | | | $g_2$ | 0.350 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.070 | |
| | | | $g_2$ | 0.030 | |
| | | $e_2$ | $g_1$ | 0.450 | |
| | | | $g_2$ | 0.450 | |

| $\Psi_5$ | | | | $P$ |
|---|---|---|---|---|
| $f_1$ | $g_1$ | $h_1$ | 0.2 | |
| | | $h_2$ | 0 | |
| | $g_2$ | $h_1$ | 0.5 | |
| | | $h_2$ | 0 | |
| $f_2$ | $g_1$ | $h_1$ | 0.4 | |
| | | $h_2$ | 0 | |
| | $g_2$ | $h_1$ | 0.7 | |
| | | $h_2$ | 0 | |

# Example: Step 4: Evidence $H = h_1$ (Sending Messages)




| $\Psi_2$ | | | | $P$ |
|---|---|---|---|---|
| $a_1$ | $b_1$ | $c_1$ | 0.036 | |
| | | $c_2$ | 0.084 | |
| | $b_2$ | $c_1$ | 0.144 | |
| | | $c_2$ | 0.336 | |
| $a_2$ | $b_1$ | $c_1$ | 0.028 | |
| | | $c_2$ | 0.012 | |
| | $b_2$ | $c_1$ | 0.252 | |
| | | $c_2$ | 0.108 | |

| $\Psi_4$ | | | $P$ |
|---|---|---|---|
| $b_1$ | $d_1$ | 0.4 | |
| | $d_2$ | 0.6 | |
| $b_2$ | $d_1$ | 0.7 | |
| | $d_2$ | 0.3 | |

| $\Psi_3$ | | | | $P$ |
|---|---|---|---|---|
| $c_1$ | $f_1$ | $g_1$ | 0.1 | |
| | | $g_2$ | 0.1 | |
| | $f_2$ | $g_1$ | 0.9 | |
| | | $g_2$ | 0.9 | |
| $c_2$ | $f_1$ | $g_1$ | 0.4 | |
| | | $g_2$ | 0.4 | |
| | $f_2$ | $g_1$ | 0.6 | |
| | | $g_2$ | 0.6 | |

| $\Psi_1$ | | | | | $P$ |
|---|---|---|---|---|---|
| $b_1$ | $c_1$ | $e_1$ | $g_1$ | 0.190 | |
| | | | $g_2$ | 0.010 | |
| | | $e_2$ | $g_1$ | 0.320 | |
| | | | $g_2$ | 0.480 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.380 | |
| | | | $g_2$ | 0.020 | |
| | | $e_2$ | $g_1$ | 0.240 | |
| | | | $g_2$ | 0.360 | |
| $b_2$ | $c_1$ | $e_1$ | $g_1$ | 0.210 | |
| | | | $g_2$ | 0.090 | |
| | | $e_2$ | $g_1$ | 0.350 | |
| | | | $g_2$ | 0.350 | |
| | $c_2$ | $e_1$ | $g_1$ | 0.070 | |
| | | | $g_2$ | 0.030 | |
| | | $e_2$ | $g_1$ | 0.450 | |
| | | | $g_2$ | 0.450 | |

| $\Psi_5$ | | | | $P$ |
|---|---|---|---|---|
| $f_1$ | $g_1$ | $h_1$ | 0.2 | |
| | | $h_2$ | 0 | |
| | $g_2$ | $h_1$ | 0.5 | |
| | | $h_2$ | 0 | |
| $f_2$ | $g_1$ | $h_1$ | 0.4 | |
| | | $h_2$ | 0 | |
| | $g_2$ | $h_1$ | 0.7 | |
| | | $h_2$ | 0 | |

$$M_{53} = \left(\overset{f_1,g_1}{0.2}\,,\ \overset{f_1,g_2}{0.5}\,,\ \overset{f_2,g_1}{0.4}\,,\ \overset{f_2,g_2}{0.7}\right)$$

$$M_{21} = \left(\overset{b_1,c_1}{0.06}, \overset{b_1,c_2}{0.10}, \overset{b_2,c_1}{0.40}, \overset{b_2,c_2}{0.44}\right)$$

$$M_{41} = \left(\overset{b_1}{1}, \overset{b_2}{1}\right)$$

$$M_{31} = \left(\overset{c_1,g_1}{0.38}, \overset{c_1,g_2}{0.68}, \overset{c_2,g_1}{0.32}, \overset{c_2,g_2}{0.62}\right)$$

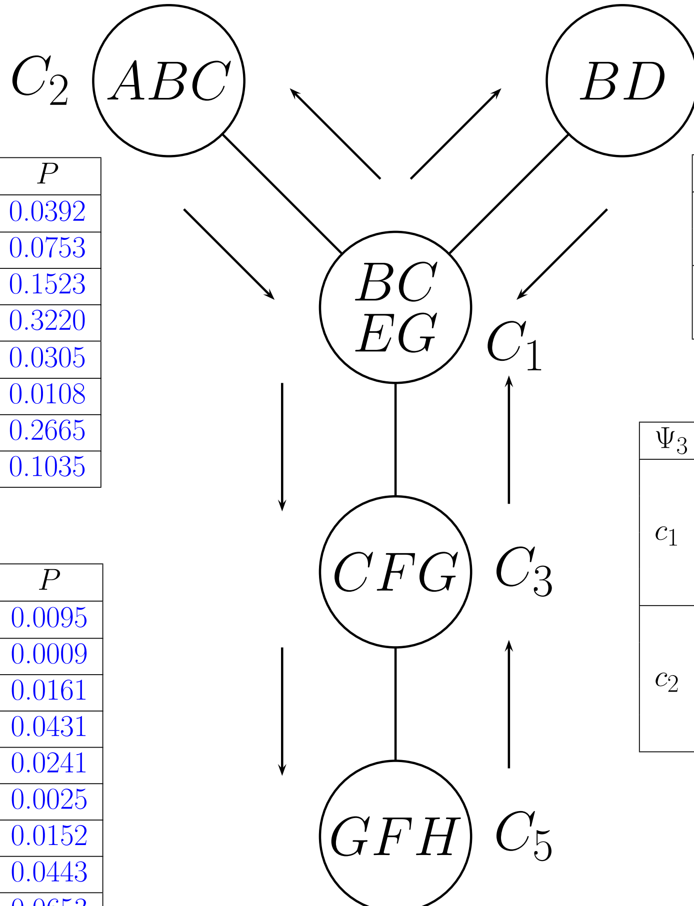
$$M_{12} = \left(\overset{b_1,c_1}{0.527}, \overset{b_1,c_2}{0.434}, \overset{b_2,c_1}{0.512}, \overset{b_2,c_2}{0.464}\right)$$

$$M_{14} = \left(\overset{b_1}{0.075}, \overset{b_2}{0.409}\right)$$

$$M_{13} = \left(\overset{c_1,g_1}{0.254}, \overset{c_1,g_2}{0.206}, \overset{c_2,g_1}{0.290}, \overset{c_2,g_2}{0.250}\right)$$

$$M_{35} = \left(\overset{f_1,g_1}{0.14}, \overset{f_1,g_2}{0.12}, \overset{f_2,g_1}{0.40}, \overset{f_2,g_2}{0.33}\right)$$

# Example: Step 4: Evidence $H = h_1$ Incorporated




| $\Psi_2$ | | | | $P$ |
|---|---|---|---|---|
| $a_1$ | $b_1$ | $c_1$ | 0.036 | 0.0392 |
| | | $c_2$ | 0.084 | 0.0753 |
| | $b_2$ | $c_1$ | 0.144 | 0.1523 |
| | | $c_2$ | 0.336 | 0.3220 |
| $a_2$ | $b_1$ | $c_1$ | 0.028 | 0.0305 |
| | | $c_2$ | 0.012 | 0.0108 |
| | $b_2$ | $c_1$ | 0.252 | 0.2665 |
| | | $c_2$ | 0.108 | 0.1035 |

| $\Psi_1$ | | | | | $P$ |
|---|---|---|---|---|---|
| $b_1$ | $c_1$ | $e_1$ | $g_1$ | 0.190 | 0.0095 |
| | | | $g_2$ | 0.010 | 0.0009 |
| | | $e_2$ | $g_1$ | 0.320 | 0.0161 |
| | | | $g_2$ | 0.480 | 0.0431 |
| | $c_2$ | $e_1$ | $g_1$ | 0.380 | 0.0241 |
| | | | $g_2$ | 0.020 | 0.0025 |
| | | $e_2$ | $g_1$ | 0.240 | 0.0152 |
| | | | $g_2$ | 0.360 | 0.0443 |
| $b_2$ | $c_1$ | $e_1$ | $g_1$ | 0.210 | 0.0653 |
| | | | $g_2$ | 0.090 | 0.0501 |
| | | $e_2$ | $g_1$ | 0.350 | 0.1088 |
| | | | $g_2$ | 0.350 | 0.1947 |
| | $c_2$ | $e_1$ | $g_1$ | 0.070 | 0.0205 |
| | | | $g_2$ | 0.030 | 0.0171 |
| | | $e_2$ | $g_1$ | 0.450 | 0.1321 |
| | | | $g_2$ | 0.450 | 0.2559 |

| $\Psi_5$ | | | | $P$ |
|---|---|---|---|---|
| $f_1$ | $g_1$ | $h_1$ | 0.2 | 0.0585 |
| | | $h_2$ | 0 | 0 |
| | $g_2$ | $h_1$ | 0.5 | 0.1243 |
| | | $h_2$ | 0 | 0 |
| $f_2$ | $g_1$ | $h_1$ | 0.4 | 0.3331 |
| | | $h_2$ | 0 | 0 |
| | $g_2$ | $h_1$ | 0.7 | 0.4841 |
| | | $h_2$ | 0 | 0 |

| $\Psi_4$ | | | $P$ |
|---|---|---|---|
| $b_1$ | $d_1$ | 0.4 | 0.0623 |
| | $d_2$ | 0.6 | 0.0934 |
| $b_2$ | $d_1$ | 0.7 | 0.5910 |
| | $d_2$ | 0.3 | 0.2533 |

| $\Psi_3$ | | | | $P$ |
|---|---|---|---|---|
| $c_1$ | $f_1$ | $g_1$ | 0.1 | 0.0105 |
| | | $g_2$ | 0.1 | 0.0212 |
| | $f_2$ | $g_1$ | 0.9 | 0.1892 |
| | | $g_2$ | 0.9 | 0.2675 |
| $c_2$ | $f_1$ | $g_1$ | 0.4 | 0.0480 |
| | | $g_2$ | 0.4 | 0.1031 |
| | $f_2$ | $g_1$ | 0.6 | 0.1440 |
| | | $g_2$ | 0.6 | 0.2165 |

$$M_{53} = (\overset{f_1,g_1}{0.2}, \overset{f_1,g_2}{0.5}, \overset{f_2,g_1}{0.4}, \overset{f_2,g_2}{0.7})$$

$$M_{21} = (\overset{b_1,c_1}{0.06}, \overset{b_1,c_2}{0.10}, \overset{b_2,c_1}{0.40}, \overset{b_2,c_2}{0.44})$$

$$M_{41} = (\overset{b_1}{1}, \overset{b_2}{1})$$

$$M_{31} = (\overset{c_1,g_1}{0.38}, \overset{c_1,g_2}{0.68}, \overset{c_2,g_1}{0.32}, \overset{c_2,g_2}{0.62})$$

$$M_{12} = (\overset{b_1,c_1}{0.527}, \overset{b_1,c_2}{0.434}, \overset{b_2,c_1}{0.512}, \overset{b_2,c_2}{0.464})$$

$$M_{14} = (\overset{b_1}{0.075}, \overset{b_2}{0.409})$$

$$M_{13} = (\overset{c_1,g_1}{0.254}, \overset{c_1,g_2}{0.206}, \overset{c_2,g_1}{0.290}, \overset{c_2,g_2}{0.250})$$

$$M_{35} = (\overset{f_1,g_1}{0.14}, \overset{f_1,g_2}{0.12}, \overset{f_2,g_1}{0.40}, \overset{f_2,g_2}{0.33})$$

| $P$ | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
|---|---|---|---|---|---|---|---|---|
| $\cdot_1$ | **0.5888** | **0.1557** | **0.4884** | **0.6533** | **0.1899** | **0.1828** | **0.3916** | **1.0000** |
| $\cdot_2$ | **0.4112** | **0.8443** | **0.5116** | **0.3467** | **0.8101** | **0.8172** | **0.6084** | **0.0000** |

# Summary

There are several exact inference methods such as variable elimination, clique tree propagation or recursive conditioning. These algorithms have complexity that is exponential with networks tree width. Exact inference is NP-hard.

In very large applications it is necessary to introduce topological structural constraints or restrictions on conditional probabilities, i.e. bounded variance algorithms.