## Decision Theory

## Descriptive Decision Theory

Descriptive Decision Theorie tries to simulate human behavior in finding the right or best decision for a given problem

Example:

- Company can chose one of two places for a new store
- Option 1: 125.000 EUR profit per year
- Option 2: 150.000 EUR profit per year

Company should take Option 2, because it maximized the profit.
Often, there are multiple target values, which should be optimal
Example (additional Information):

- Option 1: 2.000.000 EUR sales per year
- Option 2: 1.800.000 EUR sales per year

There is a conflict to handle

## Decisions under Uncertainty

In real world not every thing is known, so there are uncertainties in the model
Example:

- There are plans for restructure the local traffic, which changes the predicted profit
- Option 1: 125.000 EUR profit per year
- Option 2: 80.000 EUR profit per year

With modification Option 1 is the better one and without modification Option 2 is the better one

To model these variations in the environment we use so called Decision Tables

|  | $z_{1}$ (no modification) | $z_{2}$ (restructure) |
| :--- | ---: | ---: |
| $a_{1}$ (Option 1) | $125.000=e_{11}$ | $125.000=e_{12}$ |
| $a_{2}$ (Option 2) | $150.000=e_{21}$ | $80.000=e_{22}$ |

## Probability-based Decisions

In many cases probabilities could be assigned to each option
Objective Probabilities based on mathematic or statistic background
Subjective Probabilities based on intuition or estimations
Example:

- The management estimates the probability for the restructure to $30 \%$ The decision can be chosen by expectation value

|  | $z_{1}$ (no modification) | $z_{2}$ (restructure) | Expectation Value |
| :--- | ---: | ---: | ---: |
|  | $p_{1}=0.7$ | $p_{2}=0.3$ |  |
| $a_{1}$ (Option 1) | $125.000=e_{11}$ | $125.000=e_{12}$ | 125.000 |
| $a_{2}$ (Option 2) | $150.000=e_{21}$ | $80.000=e_{22}$ | 129.000 |

Option 2 has the higher expectation value and should be used

## Domination

An alternative $a_{1}$ dominates $a_{2}$ if the value of $a_{1}$ is always greater of (or equal to) the value of $a_{2}$

$$
{ }_{j} e_{1 j} \geq e_{2 j}
$$

Example:

|  | $z_{1}$ | $z_{2}$ |
| ---: | ---: | ---: |
| $a_{1}$ | $150.000=e_{11}$ | $90.000=e_{12}$ |
| $a_{2}$ | $125.000=e_{21}$ | $80.000=e_{22}$ |

Alternative $a_{2}$ could be dropped

## Domination - Example 2

Some more alternatives:

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ | $z_{5}$ |  |
| :--- | ---: | ---: | ---: | ---: | :--- | :--- |
| $a_{1}$ | 0 | 20 | 10 | 60 | 25 | dominated by $a_{3}$ |
| $a_{2}$ | -20 | 80 | 10 | 10 | 60 |  |
| $a_{3}$ | 20 | 60 | 20 | 60 | 50 |  |
| $a_{4}$ | 55 | 40 | 60 | 10 | 40 |  |
| $a_{5}$ | 50 | 10 | 30 | 5 | 20 | dominated by $a_{4}$ |

- $a_{3}$ dominated $a_{1}$
- $a_{4}$ dominated $a_{5}$

Alternatives $a_{1}$ and $a_{5}$ could be dropped

## Probability Domination

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ |
| ---: | ---: | ---: | ---: | ---: |
|  | $p_{1}=0.3$ | $p_{2}=0.2$ | $p_{1}=0.4$ | $p_{2}=0.1$ |
| $a_{1}$ | 20 | 40 | 10 | 50 |
| $a_{2}$ | 60 | 30 | 50 | 20 |

Probability Domination means that the cumulated probability for the payout for is always higher

## Algorithm:

- Order payout by value in a decreasing order
- Cumulate probabilities


## Example:

- $a_{1}: 50(0.1) \quad 40(0.2) \quad 20(0.3) \quad 10(0.4)$
- $a_{2}: ~ 60(0.3) \quad 50(0.4) \quad 30(0.2) \quad 20(0.1)$


## Probability Domination

## Example:

- $a_{1}: 50(0.1) \quad 40(0.2) \quad 20(0.3) \quad 10(0.4)$
- $a_{2}: \quad 60(0.3) \quad 50(0.4) \quad 30(0.2) \quad 20(0.1)$

$a_{2}$ dominates $a_{1}$.


## Multi Criteria Decisions

Optimization for multiple Targets

Complementary Targets

- Selling left foot shoes / Selling right foot shoes
- One could be avoided

Independent Targets

- Could be optimized separately

Competitive Targets

- Increase profit and sales
- Decrease environment pollution


## Multi Criteria Decisions - Example

|  | Price | Sales $e_{1}$ | Profit $e_{2}$ | Environment Pollution $e_{3}$ |
| ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | 15 | 800 | 7000 | -4 |
| $a_{2}$ | 20 | 600 | 7000 | -2 |
| $a_{3}$ | 25 | 400 | 6000 | 0 |
| $a_{4}$ | 30 | 200 | 4000 | 0 |

## Efficient Alternatives

- Only focus on alternatives which are not dominated by others
- Example: Drop $a_{4}$


## Finding a decision

- If multiple alternatives are effective we need an algorithm to choose the preferred one
- Simplest algorithm: Chose one target (most important, alphabetical) and optimize for this value


## Multi Criteria Decisions - Utility Function

Goal find a function $U\left(e_{1}, e_{2}, \ldots, e_{n}\right)$ as a combination of all targets, which could be optimized

## Linear combination

- Simplest variant: Linear combination of all targets
- $U\left(e_{1}, e_{2}, \ldots, e_{i}\right)=\sum_{i=1}^{n} \omega_{i} \cdot e_{i}$


## Example

$$
\circ \omega_{1}=10, \quad \omega_{2}=1, \quad \omega_{3}=500
$$

|  | Price | Sales $e_{1}$ | Profit $e_{2}$ | Environment Pollution $e_{3}$ | $U\left(e_{1}, e_{2}, e_{3}\right)$ |
| ---: | ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | 15 | 800 | 7000 | -4 | $\mathbf{1 3 0 0 0}$ |
| $a_{2}$ | 20 | 600 | 7000 | -2 | 12000 |
| $a_{3}$ | 25 | 400 | 6000 | 0 | 10000 |

## Decision under Uncertainty

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ |
| ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | 60 | 30 | 50 | 60 |
| $a_{2}$ | 10 | 10 | 10 | 140 |
| $a_{3}$ | -30 | 100 | 120 | 130 |

Think about, how you would decide!

## Decision Rules

- Maximin - Rule
- Maximax - Rule
- Hurwicz - Rule
- Savage-Niehans - Rule
- Laplace - Rule


## Maximin - Rule

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ | Minimum |
| ---: | ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | 60 | 30 | 50 | 60 | $\mathbf{3 0}$ |
| $a_{2}$ | 10 | 10 | 10 | 140 | 10 |
| $a_{3}$ | -30 | 100 | 120 | 130 | -30 |

Chose the one with the highest minimum

Contra: To pessimistic, only focus on one column

Example

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ | Minimum |
| ---: | ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | $1,000,000$ | $1,000,000$ | 0.99 | $1,000,000$ | 0.99 |
| $a_{2}$ | 1 | 1 | 1 | 1 | $\mathbf{1}$ |

## Maximax - Rule

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ | Maximum |
| ---: | ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | 60 | 30 | 50 | 60 | 60 |
| $a_{2}$ | 10 | 10 | 10 | 140 | $\mathbf{1 4 0}$ |
| $a_{3}$ | -30 | 100 | 120 | 130 | 130 |

Chose the one with the highest maximum

Contra: To optimistic, only focus on one column

Example

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ | Maximum |
| ---: | ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | $1,000,000$ | $1,000,000$ | $1,000,000$ | $1,000,000$ | $1,000,000$ |
| $a_{2}$ | $1,000,001$ | 1 | 1 | 1 | $\mathbf{1 , 0 0 0 , 0 0 1}$ |

## Hurwicz - Rule

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ | Max | Min | $\Phi\left(a_{i}\right)$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | 60 | 30 | 50 | 60 | 60 | 30 | $0.4 \cdot 60+0.6 \cdot 30=42$ |
| $a_{2}$ | 10 | 10 | 10 | 140 | 140 | 10 | $0.4 \cdot 140+0.6 \cdot 10=\mathbf{6 2}$ |
| $a_{3}$ | -30 | 100 | 120 | 130 | 130 | -30 | $0.4 \cdot 130+0.6 \cdot(-30)=34$ |

Combination of Maximin and Maximax - Rule
$\Phi(a)=\lambda \cdot \max \left(e_{i}\right)+(1-\lambda) \cdot \min \left(e_{i}\right)$
$\lambda$ represents readiness to assume risk
Contra: Only focus on two column
Example $\left(\min \left(a_{1}\right)<\min \left(a_{2}\right), \max \left(a_{1}\right)<\max \left(a_{2}\right) \Rightarrow\right.$ chose $\left.a_{2}\right)$

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ | Max | Min |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | $1,000,000$ | $1,000,000$ | $1,000,000$ | 0.99 | $1,000,000$ | 0.99 |
| $a_{2}$ | $1,000,001$ | 1 | 1 | 1 | $\mathbf{1 , 0 0 0 , 0 0 1}$ | $\mathbf{1}$ |

## Savage-Niehans - Rule

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ |
| ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | $\mathbf{6 0}$ | 30 | 50 | 60 |
| $a_{2}$ | 10 | 10 | 10 | $\mathbf{1 4 0}$ |
| $a_{3}$ | -30 | $\mathbf{1 0 0}$ | $\mathbf{1 2 0}$ | 130 |

Rule of minimal regret

## Algorithm:

- Find the maximal value for every column
- Subtract value from maximal value
- Use alternative with the lowest regret

Regret Table:

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ | $\operatorname{Max}$ |
| ---: | ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | $60-60=0$ | 70 | 70 | 80 | $\mathbf{8 0}$ |
| $a_{2}$ | $60-10=50$ | 90 | 110 | 0 | 110 |
| $a_{3}$ | $60-(-30)=90$ | 0 | 0 | 10 | 90 |

## Savage-Niehans - Rule II

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ |
| ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | 1,000 | $1,000,000$ | $1,000,000$ | $1,000,000$ |
| $a_{2}$ | 1,001 | 0 | 0 | 0 |

Another example
we chose $a_{1}$

Regret Table:

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ | $\operatorname{Max}$ |
| ---: | ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | 1 | 0 | 0 | 0 | $\mathbf{1}$ |
| $a_{2}$ | 0 | $1,000,000$ | $1,000,000$ | $1,000,000$ | $1,000,000$ |

## Savage-Niehans - Rule III

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ |
| ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | 1,000 | $1,000,000$ | $1,000,000$ | $1,000,000$ |
| $a_{2}$ | 1,001 | 0 | 0 | 0 |
| $a_{3}$ | $2,000,000$ | $-1,000,000$ | $-1,000,000$ | $-1,000,000$ |

Same example, but we add alternative $a_{3}$

Now we chose $a_{2}$

Regret Table:

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ | $\operatorname{Max}$ |
| ---: | ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | $1,999,000$ | 0 | 0 | 0 | $1,999,000$ |
| $a_{2}$ | $1,998,999$ | $1,000,000$ | $1,000,000$ | $1,000,000$ | $\mathbf{1 , 9 9 8}, \mathbf{9 9 9}$ |
| $a_{2}$ | 0 | $2,000,000$ | $2,000,000$ | $2,000,000$ | $2,000,000$ |

## Savage-Niehans - Rule IV

## What this means in real life:

- Student think about swimming $a_{1}$ and running $a_{2}$
- The fun factor is depending on the weather $z_{1} \ldots z_{4}$
- Student decides to go swimming
- He talk to a friend and presents his plans for the evening
- The friend mentioned to go for a $\mathrm{BBQ} a_{3}$
- With the option for BBQ the student decides to go running


## Laplace - Rule

|  | $z_{1}$ | $z_{2}$ | $z_{3}$ | $z_{4}$ | Mean |
| ---: | ---: | ---: | ---: | ---: | ---: |
| $a_{1}$ | 60 | 30 | 50 | 60 | 50 |
| $a_{2}$ | 10 | 10 | 10 | 140 | 42.5 |
| $a_{3}$ | -30 | 100 | 120 | 130 | $\mathbf{8 0}$ |

Chose the one with the highest mean value

## Contra:

- Not every condition has the same probability
- Duplication of one condition could change the result

Most people would also chose $a_{3}$ in this example

## Rule - Axioms

The following axioms should be fulfilled by the rules

## Addition to a column

The decision should not be changed, if a fixed value is added to a column

## Additional rows

The preference relation between two alternatives should not be changed, if a new row is added

## Domination

If $a_{1}$ dominates $a_{2}, a_{2}$ could not be optimal

## Join of equal columns

The preference relation between to alternatives should not change, if two columns with the same outcomes are joined to a common column

## Decision Rules Conclusion

| Rule | Example <br> Result | Addition <br> to a row | Additional <br> Rows | Domination | Join of <br> equal Rows |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Maximin | $a_{1}$ |  | $\sqrt{ }$ |  | $\sqrt{~}^{\prime}$ |
| Maximax | $a_{2}$ |  | $\sqrt{ }$ |  | $\sqrt{ }$ |
| Hurwicz | $a_{2}$ |  | $\sqrt{ }$ |  | $\sqrt{ }$ |
| Savage-Niehans | $a_{1}$ | $\sqrt{ }$ |  | $\sqrt{ }$ | $\sqrt{ }$ |
| Laplace | $a_{3}$ | $\sqrt{ }$ | $\sqrt{ }$ | $\sqrt{ }$ |  |

No Rule fulfills all axioms $\Rightarrow$ no perfect rule
Common usage: Remove duplicate Columns and use Laplace
Better: Define subjective probabilities and use them

## Decision Graphs / Influence Diagrams

## Preference Orderings

A preference ordering $\succeq$ is a ranking of all possible states of affairs (worlds) $S$

- these could be outcomes of actions, truth assignments, states in a search problem, etc.
- $s \succeq t$ : means that state $s$ is at least as good as $t$
- $s \succ t$ : means that state $s$ is strictly preferred to $t$

We insist that $\succeq$ is

- reflexive: i.e., $\mathrm{s} \succeq \mathrm{s}$ for all states s
- transitive: i.e., if $\mathrm{s} \succeq \mathrm{t}$ and $\mathrm{t} \succeq \mathrm{w}$, then $\mathrm{s} \succeq \mathrm{w}$
- connected: for all states $\mathrm{s}, \mathrm{t}$, either $\mathrm{s} \succeq \mathrm{t}$ or $\mathrm{t} \succeq \mathrm{s}$


## Why Impose These Conditions?

Structure of preference ordering imposes certain "rationality requirements" (it is a weak ordering)
E.g., why transitivity?

- Suppose you (strictly) prefer coffee to tea, tea to OJ, OJ to coffee
- If you prefer X to Y , you will trade me Y plus $\$ 1$ for X
- I can construct a "money pump" and extract arbitrary amounts of money from you


## Utilities

Rather than just ranking outcomes, we must quantify our degree of preference

- e.g., how much more important is chc than $\sim$ mess

A utility function $U: S \rightarrow \mathbb{R}$ associates a realvalued utility with each outcome.

- $U(s)$ measures your degree of preference for $s$

Note: $U$ induces a preference ordering $\succeq_{U}$ over S defined as: $\mathrm{s} \succeq_{U} \mathrm{t}$ iff $U(s) \geq$ $U(t)$

- obviously $\succeq_{U}$ will be reflexive, transitive, connected


## Expected Utility

Under conditions of uncertainty, each decision d induces a distribution $P r_{d}$ over possible outcomes

- $\operatorname{Pr}_{d}(s)$ is probability of outcome s under decision d

The expected utility of decision d is defined
The principle of maximum expected utility (MEU) states that the optimal decision under conditions of uncertainty is that with the greatest expected utility.

$$
E U(d)=\sum_{s \in S} \operatorname{Pr}_{d}(s) U(s)
$$

## Decision Problems: Uncertainty

A decision problem under uncertainty is:

- a set of decisions D
- a set of outcomes or states S
- an outcome function $\operatorname{Pr}: D \rightarrow \Delta(S)$
* $\Delta(S)$ is the set of distributions over S (e.g., Prd)
- a utility function U over S

A solution to a decision problem under uncertainty is any $d^{*} \in D$ such that $E U\left(d^{*}\right) \succeq E U(d)$ for all $d \in D$

Again, for single-shot problems, this is trivial

## Expected Utility: Notes

Note that this viewpoint accounts for both:

- uncertainty in action outcomes
- uncertainty in state of knowledge
- any combination of the two


Stochastic actions


Uncertain knowledge

## Expected Utility: Notes

Why MEU? Where do utilities come from?

- underlying foundations of utility theory tightly couple utility with action/choice
- a utility function can be determined by asking someone about their preferences for actions in specific scenarios (or "lotteries" over outcomes)

Utility functions needn't be unique

- if I multiply U by a positive constant, all decisions have same relative utility
- if I add a constant to U, same thing
- $U$ is unique up to positive affine transformation


## So What are the Complications?

Outcome space is large

- like all of our problems, states spaces can be huge
- don't want to spell out distributions like $P r_{d}$ explicitly
- Solution: Bayes nets (or related: influence diagrams)

Decision space is large

- usually our decisions are not one-shot actions
- rather they involve sequential choices (like plans)
- if we treat each plan as a distinct decision, decision space is too large to handle directly
- Soln: use dynamic programming methods to construct optimal plans (actually generalizations of plans, called policies... like in game trees)


## So What are the Complications?

Decision networks (more commonly known as influence diagrams) provide a way of representing sequential decision problems

- basic idea: represent the variables in the problem as you would in a BN
- add decision variables - variables that you "control"
- add utility variables - how good different states are


## Sample Decision Network



## Decision Networks: Chance Nodes

Chance nodes

- random variables, denoted by circles
- as in a BN, probabilistic dependence on parents



## Decision Networks: Decision Nodes

## Decision nodes

- variables decision maker sets, denoted by squares
- parents reflect information available at time decision is to be made

In example decision node: the actual values of Ch and Fev will be observed before the decision to take test must be made

- agent can make different decisions for each instantiation of parents (i.e., policies)


$$
\text { BT } \epsilon\{b t, \sim b t\}
$$

## Decision Networks: Decision Nodes

Value node

- specifies utility of a state, denoted by a diamond
- utility depends only on state of parents of value node
- generally: only one value node in a decision network

Utility depends only on disease and drug

$U($ fludrug, flu) $=20$
$U($ fludrug, mal $)=-300$
$U($ fludrug, none $)=-5$
$U($ maldrug, flu $)=-30$
$U($ maldrug, mal$)=10$
$U$ (maldrug, none) $=-20$
$U($ no drug, flu) $=-10$
$U($ no drug, mal $)=-285$
$U($ no drug, none $)=30$

## Decision Networks: Assumptions

Decision nodes are totally ordered

- decision variables $D_{1}, D_{2}, \ldots, D_{n}$
- decisions are made in sequence
- e.g., BloodTst (yes,no) decided before Drug (fd,md,no)

No-forgetting property

- any information available when decision $D_{i}$ is made is available when decision $D_{j}$ is made (for $i<j$ )
- thus all parents of $D_{i}$ are parents of $D_{j}$


Dashed arcs ensure the no-forgetting property

## Policies

Let $\operatorname{Par}\left(D_{i}\right)$ be the parents of decision node $D_{i}$

- $\operatorname{Dom}(\operatorname{Par}(D i))$ is the set of assignments to parents

A policy $\delta$ is a set of mappings $\delta_{i}$, one for each decision node $D_{i}$

- $\delta_{i}: \operatorname{Dom}\left(\operatorname{Par}\left(D_{i}\right)\right) \rightarrow\left(D_{i}\right)$
- $\delta_{i}$ associates a decision with each parent assignment for $D_{i}$

For example, a policy for BT might be:

$$
\begin{aligned}
\delta_{B T}(c, f) & =b t \\
\delta_{B T}(c, \sim f) & =\sim b t \\
\delta_{B T}(\sim c, f) & =b t \\
\delta_{B T}(\sim c, \sim f) & =\sim b t
\end{aligned}
$$



## Value of a Policy

Value of a policy $\delta$ is the expected utility given that decision nodes are executed according to $\delta$

Given associates $\boldsymbol{x}$ to the set $\boldsymbol{X}$ of all chance variables, let $\delta(\boldsymbol{x})$ denote the assignment to decision variables dictated by $\delta$

- e.g., assigned to $D_{1}$ determined by it's parents' assignment in $\boldsymbol{x}$
- e.g., assigned to $D_{2}$ determined by it's parents' assignment in $\boldsymbol{x}$ along with whatever was assigned to D1
- etc.

Value of $\delta$ :

$$
E U(\delta)=\sum_{\boldsymbol{X}} P(\boldsymbol{X}, \delta(\boldsymbol{X}) U(\boldsymbol{X}, \delta(\boldsymbol{X}))
$$

## Optimal Policies

An optimal policy is a policy $\delta^{*}$ such that $E U\left(\delta^{*}\right) \geq E U(\delta)$ for all policies $\delta$
We can use the dynamic programming principle yet again to avoid enumerating all policies

We can also use the structure of the decision network to use variable elimination to aid in the computation

## Computing the Best Policy

We can work backwards as follows
First compute optimal policy for Drug (last dec'n)

- for each assignment to parents (C,F,BT,TR) and for each decision value ( $\mathrm{D}=$ $\mathrm{md}, \mathrm{fd}$,none), compute the expected value of choosing that value of D
- set policy choice for each value of parents to be the value of $D$ that has max value
- eg: $\delta_{D}(c, f, b t, p o s)=m d$



## Computing the Best Policy

Next compute policy for BT given policy $\delta_{D}(C, F, B T, T R)$ just determined for Drug

- since $\delta_{D}(C, F, B T, T R)$ is fixed, we can treat Drug as a normal random variable with deterministic probabilities
- i.e., for any instantiation of parents, value of Drug is fixed by policy $\delta_{D}$
- this means we can solve for optimal policy for BT just as before
- only uninstantiated vars are random vars (once we fix its parents)


## Computing the Best Policy

How do we compute these expected values?

- suppose we have assigned $<c, f, b t$, pos $>$ to parents of Drug
- we want to compute EU of deciding to set $\operatorname{Drug}=m d$
- we can run variable elimination!

Treat $C, F, B T, T R, D r$ as evidence

- this reduces factors (e.g., $U$ restricted to $b t, m d$ : depends on $D i s$ )
- eliminate remaining variables (e.g., only Disease left)
- left with factor: $U()=\sum_{D i s} P(D i s \mid c, f, b t, p o s, m d) U(D i s)$

We now know EU of doing $D r=m d$ when $c, f, b t$, pos true
Can do same for $f d, n o$ to decide which is best


## Computing Expected Utilities

The preceding illustrates a general phenomenon

- computing expected utilities with BNs is quite easy
- utility nodes are just factors that can be dealt with using variable elimination

$$
\begin{aligned}
E U & =\sum_{A, B, C} P(A, B, C) U(B, C) \\
& =\sum_{A, B, C} P(C \mid B) P(B \mid A) P(A) U(B, C)
\end{aligned}
$$

Just eliminate variables in the usual way


## Optimizing Policies: Key Points

If a decision node D has no decisions that follow it, we can find its policy by instantiating each of its parents and computing the expected utility of each decision for each parent instantiation

- no-forgetting means that all other decisions are instantiated (they must be parents)
- its easy to compute the expected utility using VE
- the number of computations is quite large: we run expected utility calculations (VE) for each parent instantiation together with each possible decision D might allow
- policy: choose max decision for each parent instant'n


## Optimizing Policies: Key Points

When a decision D node is optimized, it can be treated as a random variable

- for each instantiation of its parents we now know what value the decision should take
- just treat policy as a new CPT: for a given parent instantiation $\boldsymbol{x}$, D gets $\delta(\boldsymbol{x})$ with probability 1 (all other decisions get probability zero)

If we optimize from last decision to first, at each point we can optimize a specific decision by (a bunch of) simple VE calculations

- it's successor decisions (optimized) are just normal nodes in the BNs (with CPTs)


## Decision Network Notes

Decision networks commonly used by decision analysts to help structure decision problems

Much work put into computationally effective techniques to solve these

- common trick: replace the decision nodes with random variables at outset and solve a plain Bayes net (a subtle but useful transformation)

Complexity much greater than BN inference

- we need to solve a number of BN inference problems
- one BN problem for each setting of decision node parents and decision node value


## Decision Network Notes

In example on previous slide:

- we assume the state (of the variables at any stage) is fully observable * hence all time $t$ vars point to time $t$ decision
- this means the state at time t d-separates the decision at time $\mathrm{t}-1$ from the decision at time t-2
- so we ignore "no-forgetting" arcs between decisions
* once you know the state at time t , what you did at time $\mathrm{t}-1$ to get there is irrelevant to the decision at time $\mathrm{t}-1$

If the state were not fully observable, we could not ignore the "no-forgetting" arcs

## A Detailed Decision Net Example

Setting: you want to buy a used car, but there's a good chance it is a "lemon" (i.e., prone to breakdown). Before deciding to buy it, you can take it to a mechanic for inspection. S/he will give you a report on the car, labelling it either "good" or "bad". A good report is positively correlated with the car being sound, while a bad report is positively correlated with the car being a lemon.

The report costs $\$ 50$ however. So you could risk it, and buy the car without the report.

Owning a sound car is better than having no car, which is better than owning a lemon.

## Car Buyer's Network



## Evaluate Last Decision: Buy (1)

$$
\begin{aligned}
& E U(B \mid I, R)=\sum_{L} P(L \mid I, R, B) U(L, B) \\
& \begin{aligned}
& I=i, R=g: \\
& E U(\text { buy })=P(l \mid i, g) U(l, b u y)+P(\sim l \mid i, g) U(\sim l, \text { buy })-50 \\
&=.18 \cdot-600+.82 \cdot 1000-50=662 \\
& E U(\sim \text { buy })=P(l \mid i, g) U(l, \sim b u y)+P(\sim l \mid i, g) U(\sim l, \sim b u y)-50 \\
&=-300-50=-350(-300 \text { indep. of lemon })
\end{aligned}
\end{aligned}
$$

So optimal $\delta_{B u y}(i, g)=b u y$

## Evaluate Last Decision: Buy (2)

$$
\begin{aligned}
& I=i, R=b: \\
& \\
& E U(\text { buy })
\end{aligned} \begin{aligned}
& =P(l \mid i, b) U(l, \text { buy })+P(\sim l \mid i, b) U(\sim l, \text { buy })-50 \\
& =.89 \cdot-600+.11 \cdot 1000-50=-474 \\
E U(\sim \text { buy }) & =P(l \mid i, b) U(l, \sim b u y)+P(\sim l \mid i, b) U(\sim l, \sim b u y)-50 \\
& =-300-50=-350(-300 \text { indep. of lemon })
\end{aligned}
$$

So optimal $\delta_{\text {Buy }}(i, b)=\sim$ buy

## Evaluate Last Decision: Buy (3)

$I=\sim i, R=g$ (note: no inspection cost subtracted):

$$
\begin{aligned}
E U(\text { buy }) & =P(l \mid \sim i, g) U(l, \text { buy })+P(\sim l \mid \sim i, g) U(\sim l, \text { buy }) \\
& =.5 \cdot-600+.5 \cdot 1000=200 \\
E U(\sim \text { buy }) & =P(l \mid \sim i, g) U(l, \sim \text { buy })+P(\sim l \mid \sim i, g) U(\sim l, \sim b u y)-50 \\
& =-300-50=-350(-300 \text { indep. of lemon })
\end{aligned}
$$

So optimal $\delta_{B u y}(\sim i, g)=\sim$ buy
So optimal policy for Buy is:

- $\delta_{B u y}(i, g)=$ buy; $\delta_{B u y}(i, b)=\sim$ buy; $\delta_{B u y}(\sim i, n)=b u y$

Note: we don't bother computing policy for $(i, \sim n)$, $(\sim i, g)$, or $(\sim i, b)$, since these occur with probability 0

## Evaluate First Decision: Inspect

$E U(I)=\sum_{L, R} P(L, R \mid I) U\left(L, \boldsymbol{\delta}_{\boldsymbol{B u y}}(\boldsymbol{I}, \boldsymbol{R})\right)$,
where $P(R, L \mid I)=P(R \mid L, I) P(L \mid I)$

$$
\begin{aligned}
E U(i) & =.1 \cdot-600+.4 \cdot-300+.45 \cdot 1000+.05 \cdot-300-50 \\
& =237.5-50=187.5 \\
E U(\sim i) & =P(l \mid \sim i, n) U(l, \text { buy })+P(\sim l \mid \sim i, n) U(\sim l, \text { buy }) \\
& =.5 \cdot-600+.5 \cdot 1000=200
\end{aligned}
$$

So optimal $\delta_{\text {Inspect }}(\sim i)=$ buy

|  | $P(R, L \mid I)$ | $\boldsymbol{\delta}_{\boldsymbol{B u y}}$ | $U\left(L, \boldsymbol{\delta}_{\boldsymbol{B u y}}\right)$ |
| :--- | :--- | :--- | :--- |
| $g, l$ | 0.1 | buy | $-600-50=-650$ |
| $g, \sim l$ | 0.45 | buy | $1000-50=950$ |
| $b, l$ | 0.4 | $\sim$ buy | $-300-50=-350$ |
| $b, \sim l$ | 0.05 | $\sim$ buy | $-300-50=-350$ |

## Value of Information

So optimal policy is: don't inspect, buy the car

- $\mathrm{EU}=200$
- Notice that the EU of inspecting the car, then buying it iff you get a good report, is 237.5 less the cost of the inspection (50). So inspection not worth the improvement in EU.
- But suppose inspection cost $\$ 25$ : then it would be worth it $(E U=237.5-25=$ $212.5>E U(\sim i))$
- The expected value of information associated with inspection is 37.5 (it improves expected utility by this amount ignoring cost of inspection). How? Gives opportunity to change decision ( $\sim$ buy if bad).
- You should be willing to pay up to $\$ 37.5$ for the report

Slide of this section were taken from CSC 384 Lecture Slides © $2002-2003$, C. Boutilier and P. Poupart

